The Standard Theory of Cosmology

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Introduction

The physical science called "cosmology" means physics in extremely large scale, in such a huge scale that galaxies and also hoops of galaxies are treated as a more or less continuous distribution of dust. One major part of cosmology is the space-time expansion, which obeys under the geometrical laws of general relativity. The other major part is particle physics in cosmological scale. In this article the focus is in the geometrical part and general relativity.

In general relativity one of the main results is what is called the cosmological field equation. In effect it is a Poisson equation of the whole universe, of the total gravitational field of all the masses of the universe. The usual way deducing it is starting from a general expression of differential length in spacetime geometry, which is called the line element.

The mathematical expression of space-time geometry in the Big-Bang scenario is what is called a Friedmann-Lemaitre-Robertson-Walker (FLRW) line element. There is a controversy of who was the first to suggest this geometrical solution to Einstein's equations, so the compromise has became referencing to them all. As we see, it is a very diversified troop of scientists and gentlemen:

- Alexander Friedmann (1888 - 1925) was a Russian mathematician and physicist. He participated to the World War I and lived through the Russian revolution. He was born in St. Petersburg and became a professor in the university there, and was later professor in Perm State University, one of the more renowned in USSR (Soviet). One of his students was Georg Gamow.

- Georges Henri Joseph Édouard Lemaître (1894 - 1966) was a Belgian mathematician and physicist. He was also a catholic priest, and is often referenced as Monsignor or Abbé Lemaître. He was professor at the Catholic University of Louvain, and became a member and later the president of the Pontifical Academy of Sciences (the academy of the Vatican; that may be the reason why Pope Pius XII in 1951 canonized the Big Bang theory.)

- Howard Percy Robertson (1903 - 1961) was an American mathematician and physicist. He was born in Hoquiam in Washington state, and worked in many universities in USA. He was also employed by a couple of American intelligence agencies.

- Arthur Geoffrey Walker (1909 - 2001) was an English mathematician and cosmologist, born in Watford, Hertfordshire, England. He doctorated in University of Edinburgh, and worked in universities of Sheffield and Liverpool. He also co-worked with the Italian physicist Enrico Fermi.

The FLRW Line Element

We start developing it from a general form of space-time line-element:

$$c^2\,d\tau^2 = c^2\,dt^2 - d\Sigma^2 \tag{1}$$

Often the proper time is replaced by s = ct, and thus ds^2 in stead of $c^2\ d\tau^2$ on the left side. On the right $d\Sigma^2$ is a spatial (3-dimesional) line-element. Expressed in spherical coordinates R, θ, and ϕ, this becomes

$$d\Sigma^2 = dR^2 + R^2\ d\theta^2 + R^2\ \sin(\theta)^2\ d\phi^2 = dR^2 + R^2\ d\Omega^2 \tag{2}$$

where R is the 3-dim. radial coordinate the angular part is denoted:

$$d\Omega^2 = d\theta^2 + \sin(\theta)^2\ d\phi^2 \tag{3}$$

If the space is not Euclidean, the curvature radius a is defined as the radius of an osculating sphere, that "so closely as possible" touches the surface without intersecting it. It is a mathematical definition in 3-dimensional algebra, and has nothing to do with relativity. The curvature K is defined as follows:

$$\begin{aligned} K &= \frac{1}{a^2} \quad \text{for positive curvature,} \\ K &= -\frac{1}{a^2} \quad \text{for negative curvature} \end{aligned} \tag{4}$$

K = 0 for non-curved 3-space. Positive curvature has surfaces such as an ellipsoid, and negative curvature such as a saddle surface. In commonplace 3-space geometry a curved-space line-element is:

$$d\Sigma^2 = \frac{dR^2}{1 - K\ R^2} + R^2\ d\Omega^2 \tag{5}$$

We then introduce a dimensionless radial variable:

$$r = \frac{R}{a} \tag{6}$$

and write (4) in form

$$K = \frac{k}{a^2} \tag{7}$$

where k = +1, –1 , or 0 . Then using (4) we get from (5):

$$d\Sigma^2 = a^2 \left(\frac{dr^2}{1 - k\ r^2} + r^2\ d\Omega^2 \right) \tag{8}$$

The scale factor a now alone gives the curvature rate, and k only gives if it is positive (k = +1), negative (k = –1) or if there is no curvature at all (k = 0). In the last case the scale factor only affects the scale of the space but not the curvature. Because we assumed spatial symmetry the scale factor a can not depend on the spatial coordinates, but we can let it be time depended. So we conclude that the most general form of spatially symmetric curved space-time line-element is the one called FLRW line-element:

$$c^2\ d\tau^2 = c^2\ dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2\ d\theta^2 + r^2\ \sin(\theta)^2\ d\phi^2 \right) \tag{9}$$

The Metric Tensor in FLRW-metric

The metric tensor now becomes

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-a(t)^2}{1-k\,r^2} & 0 & 0 \\ 0 & 0 & -a(t)^2\,r^2 & 0 \\ 0 & 0 & 0 & -a(t)^2\,r^2\,\sin(\theta)^2 \end{pmatrix} \tag{10}$$

The 4-volume element is defined

$$dQ = \sqrt{-\det(g)}\;du_1\,...du_n \tag{11}$$

Determinant of g is here

$$\det(g) = \frac{-a(t)^2}{1-k\,r^2}\left(-a(t)^2\,r^2\right)\left(-a(t)^2\,r^2\,\sin(\theta)^2\right) = -\frac{a(t)^6}{1-k\,r^2}\,r^4\,\sin(\theta)^2 \tag{12}$$

So the curved 4-space 4-volume element becomes

$$dQ = \frac{a(t)^3}{\sqrt{1-k\,r^2}}\cdot r^2\,\sin(\theta)\;dt\;dr\;d\theta\;d\phi \tag{13}$$

and a 3-volume element, that has all but the first diagonal element:

$$dV = \frac{a(t)^3}{\sqrt{1-k\,r^2}}\cdot r^2\,\sin(\theta)\;dt\;dr\;d\theta\;d\phi \tag{14}$$

Connection coefficients

The connection coefficients, expressed as Christoffel symbols of second kind, become

$$\Gamma^{\sigma}_{\mu\nu} = \frac{g^{\rho\sigma}}{2}\left(\frac{\partial g_{\nu\rho}}{\partial q^{\mu}} + \frac{\partial g_{\mu\rho}}{\partial q^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial q^{\rho}}\right) \tag{15}$$

It is practical to write them as matrices, especially for computer aided symbolic calculations. They make four 2x2 matrices, the first Christoffel symbol index gives the particular matrix, and the other two indices the corresponding matrix component. These formal symbolic matrices then form a pseudovector, where the four components are rank 2 tensors:

$$\underline{\Gamma} = \left(\Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3\right) \tag{16}$$

The Christoffel symbols become, here expressed in the pseudovector-matrix form:

$$\Gamma^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{a(t)\; a'(t)}{1-k\, r^2} & 0 & 0 \\ 0 & 0 & a(t)\; a'(t)\; r^2 & 0 \\ 0 & 0 & 0 & a(t)\; a'(t)\; r^2\; \sin(\theta)^2 \end{pmatrix} \quad (17)$$

$$\Gamma^1 = \begin{bmatrix} 0 & \frac{a'(t)}{a(t)} & 0 & 0 \\ \frac{a'(t)}{a(t)} & \frac{k\, r}{1-k\, r^2} & 0 & 0 \\ 0 & 0 & -r\left(1-k\, r^2\right) & 0 \\ 0 & 0 & 0 & -r\left(1-k\, r^2\right) \sin(\theta)^2 \end{bmatrix} \quad (18)$$

$$\Gamma^2 = \begin{pmatrix} 0 & 0 & \frac{a'(t)}{a(t)} & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ \frac{a'(t)}{a(t)} & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\cos(\theta)\; \sin(\theta) \end{pmatrix} \quad (19)$$

$$\Gamma^3 = \begin{pmatrix} 0 & 0 & 0 & \frac{a'(t)}{a(t)} \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ \frac{a'(t)}{a(t)} & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{pmatrix} \quad (20)$$

The Ricci Tensor

It is defined as

$$R_{\mu\nu} = g^{\delta\sigma} R_{\delta\mu\sigma\nu} = \frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial q^{\sigma}} - \frac{\partial \Gamma^{\sigma}_{\mu\sigma}}{\partial q^{\nu}} + \Gamma^{\sigma}_{\rho\sigma}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\sigma}_{\rho\nu}\Gamma^{\rho}_{\mu\sigma} \quad (21)$$

It is now diagonal, and the diagonal elements become

$$R_{00} = -\frac{3\, a''}{a} \tag{22}$$

$$R_{11} = \left(a\, a'' + 2\, a'^2 + 2\, k\right) \frac{1}{1 - k\, r^2} \tag{23}$$

$$R_{22} = \left(a\, a'' + 2\, a'^2 + 2\, k\right) r^2 \tag{24}$$

$$R_{33} = \left(a\, a'' + 2\, a'^2 + 2\, k\right) r^2 \sin(\theta)^2 \tag{25}$$

The curvature scalar is defined

$$R = g^{\mu\nu} R_{\mu\nu} = g^{00}\, R_{00} + g^{11}\, R_{11} + g^{22}\, R_{22} + g^{33}\, R_{33} \tag{26}$$

It becomes (because we have diagonal tensors):

$$R = -\frac{6}{a^2} \left(a\, a'' + a'^2 + k\right) \tag{27}$$

The Cosmological Field Equation

The Einstein tensor is defined generally as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} \tag{28}$$

Similarly as the metric tensor, it too is diagonal in the current case, and the diagonal elements become

$$G_{00} = \frac{3\left(a'^2 + k\right)}{a^2} = 3\left[\left(\frac{a'}{a}\right)^2 + \frac{k}{a^2}\right] \tag{29}$$

$$G_{11} = -a^2 \left[2\, \frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + \frac{k}{a^2}\right] \frac{1}{1 - k\, r^2} \tag{30}$$

$$G_{22} = -a^2 \left[2\, \frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + \frac{k}{a^2}\right] r^2 \tag{31}$$

$$G_{33} = -a^2 \left[2\, \frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + \frac{k}{a^2}\right] r^2 \sin(\theta)^2 \tag{32}$$

Einstein's field equation, with the cosmological constant Λ :

$$G_{\mu\nu} = \kappa\, \Gamma_{\mu\nu} + \Lambda\, g_{\mu\nu} \tag{33}$$

where $$\kappa = 8\, \pi\, G \tag{34}$$

The stress-energy tensor:

$$\Gamma_{\mu\nu} = (p + \rho)\ u_\mu\ u_\nu - p\ g_{\mu\nu} \tag{35}$$

We consider p and ρ represent the total stress-energy and pressure, respectively ($u_0 = c = 1$).

$$\Gamma_{00} = \rho \tag{36}$$

$$\Gamma_{ii} = (p + \rho)\ u_i^2 - p\ g_{ii} \qquad i = 1, 2, 3 \tag{37}$$

We get two equations that are basical for cosmology. From the 00-components we get an equation called the ***Friedmann equation*** (here with the cosmological constant included, which the original form of the Friedmann equation did not have):

$$\boxed{\left(\frac{a'}{a}\right)^2 + \frac{k}{a^2} = \frac{\kappa}{3}\ \rho + \Lambda} \tag{38}$$

And from the other three components the one called the ***acceleration equation***:

$$\boxed{2\ \frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + \frac{k}{a^2} = -\kappa\ p + \Lambda} \tag{39}$$

But generally we have in (38) and (39) two differential equations with one unknown function.

Now further, try substituting: $$u(t) = \frac{a'(t)}{a(t)} \tag{40}$$

Further, (40) gives by differentiating

$$u' = \frac{a''}{a} + \frac{a'^2}{a^2} = \frac{a''}{a} + u^2 \tag{41}$$

$$\Rightarrow \qquad \frac{a''}{a} = u' + u^2 \tag{42}$$

Eq. (38) becomes now

$$u^2 + \frac{k}{a^2} = \frac{\kappa}{3}\ \rho + \Lambda \tag{43}$$

which means that we have in effect eliminated the first differential equation, and can solve out

$$\frac{k}{a^2} = \frac{\kappa}{3}\ \rho + \Lambda - u^2 \tag{44}$$

Substituting (40), (42), and (44) to eq. (39) it becomes

$$2\left(u' + u^2\right) + u^2 + \left(\frac{\kappa}{3}\ \rho + \Lambda - u^2\right) = -\kappa\ p + \Lambda \tag{45}$$

which can be rewritten as

$$2\ u' + 2\ u^2 + \frac{\rho\ \kappa}{3} = -\kappa\ p \tag{46}$$

$$u' + u^2 = -\frac{1}{2}\left(\frac{\rho\ \kappa}{3} + \kappa\ p\right) \tag{47}$$

Solving the differential equation (47) for $u(t)$ gives us the scale factor function $a(t)$.

From (40) we get by integration:

$$\int u(t)\ dt = \ln(a(t)) \tag{48}$$

and so

$$a(t) = \exp\left(\int u(t)\ dt\right) \tag{49}$$

Note that the cosmological constant Λ is not present in the differential equation (47) above, it gets eliminated, see eq. (45). This is an expected result since it is a free constant, which can be included in the cosmological equation, but general relativity does not give any condition for its value. If it is supposed to be nonzero it must be deduced from other physical conditions, or experimentally.
