General Relativity - a Short Review

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Preface

The purpose of this text is outlining the mathematical theory of general relativity, with in fact is quite much straight forward task and not really difficult, in contrary to what rumors often tell. The only obstruction maybe is the tensor algebraic notation. But of course, the reader is expected to have some knowledge vector algebra and theory of non-Cartesian coordinate systems. Also basics of special relativity are expected to be known.

The language of general relativity is tensor algebra. The name "tensor" comes from study of solid materials under tension, an important engineering branch. If a string is set under tension the physical situation is covered by a formula called Hooke's law: the tension force is a constant times the displacement, $F = k\ d$. Though, if the tension can be put in different parts of the string, and especially then if the string is not homogeneous, the the "string constant" is not a constant any more but a function of the position x of the force vector, $k = k(x)$. Moreover, if we have a peace of solid material, in order to give the tension force all over the material we need something more elaborate to express how the material responds to the force, and that in all directions, in all space dimensions. Such an elaborate thing is called a tensor, but it is merely only a generalization of the string constant.

This I tell in order to demystify the tensor, because it is basically something very practical. It is not difficult to grasp what it is for, but it becomes mathematically intricate because it must cover all possible coordinate positions and directions of the force. A tensor has a multitude of components, arranged in certain way, and all the components are functions of the coordinates. Formally mathematically a tensor is often defined as a system of functions that transforms in a certain way under a coordinate transformation. Tensor algebra turned out to be a very potent tool, being able to express most complicated and general conditions in form of compact expressions. The compactness is much of the trouble, because tensors come with those many indices with an urge of practical training in the handiwork.

The more technological kind of science, solid state physics, or with a more modern word, condensed state physics, curiously, has turned out to be an important source of ideas for the pure theoretical physicists, offering not only ideas, but also the mathematical tools. Not only general relativity, but also more modern theories such as quantum field theory with things like Higgs particles, has borrowed ideas from condensed state physics.

1. Metric of Space-time

The main idea in the theory of General Relativity is that gravitation is an effect due to curvature of space-time. Einstein, advised by his mathematical friends, realized that the theoretical framework of tensor algebra would be suitable to express his ideas of gravitation. A tensor based theory of mathematical curved spaces were developed earlier by Riemann, and it was very much new an exiting mathematics at that time. Riemann's work was based on positively definite metric, but it was not difficult to convert the theory into the non-definite metric of Einstein - Minkowski space-time. There are two "signatures" commonly in use for the later, (-1,1,1,1) , and the opposite (1,-1,-1,-1). In special relativity, the expression of length, the "metric", is defined by the "metric tensor" of non-curved Cartesian space-time, often called Euclidean space-time:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{or} \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

We use here the later (timelike) form, but many books tend to use the former (spacelike) form in stead. It doesn't really matter, because the results will be the same in end. We denote the (spacelike) four-distance s , and the (timelike) proper time τ . We can now write

$$-s^2 = (c\ \tau)^2 = (c\ t)^2 - x^2 - y^2 - z^2 = \eta_{\mu\nu}\ q^\nu\ q^\mu \tag{2}$$

Here we have denoted the generalized coordinates (it is upper indices, not exponentials):

$$q^0 = c\,t \qquad q^1 = x \qquad q^2 = y \qquad q^3 = z \tag{3}$$

More often it is practical to consider the differential four-length ds, and corresponding differential proper time $d\tau$, and a differential quadratic form called the "line element":

$$(c\ d\tau)^2 = (c\ dt)^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}\ dq^\nu\ dq^\mu \tag{4}$$

Now going over to a curved space-time is done straight forward, simply by replacing the metric tensor with a more general one:

$$(c\ d\tau)^2 = g_{\mu\nu}\ dq^\nu\ dq^\mu \tag{5}$$

In this pure theoretical treatment we need not know the components of $g_{\mu\nu}$, only that they must be symmetrical in indices. If written as a 2x2 array and the corresponding components on both sides of the diagonal must be equal:

$$g_{\mu\nu} = g_{\nu\mu} \tag{6}$$

This denoted with the lower indices is called the "covariant" metric tensor. There is a "contravariant" form too, denoted with upper indices $g^{\mu\nu}$, and defined as the inverse of $g_{\mu\nu}$, so that their product is a unit tensor .

$$g_{\rho\nu}\ g^{\mu\nu} = \sum_\nu \left(g_{\rho\nu}\ g^{\mu\nu}\right) = \delta_\rho{}^\nu \tag{7}$$

All tensors and vectors can have covariant or contravariant indices, which indicate two contrary properties when transforming the coordinates, that is, changing a coordinate system. It has to do with the fact that tensors are arrays of functions. It is commonplace tensor algebra and we will not need to go deeper into that subject here.

Here we also have met what is called Einstein's summation convention, which makes equations shorter because the summa symbol can be leaved out. It states that where an index is repeated in a term it means summing over such an index. There is a strong and a weak rule: the weak rule says that summation is assumed in what ever position the repeated indices are, the strong rule says that summations is assumed only if the identical indices are in contra-positions, that is, one in lower, one in upper position.

2. Covariant Derivatives

An important concept in the algebra of curved spaces is the covariant derivative. It takes care of the fact that the very framework of the space is not straight but curved, and performs the differentiation accordingly. Covariant derivation of a covariant vector becomes:

$$A_{\tau;\nu} = A_{\tau,\nu} - \Gamma^\mu_{\tau\nu} A_\mu = \frac{\partial A_\tau}{\partial q^\nu} - \Gamma^\mu_{\tau\nu} A_\mu \tag{8}$$

The covariant derivative of a contravariant vector

$$A^\tau{}_{;\nu} = A^\tau{}_{,\nu} + \Gamma^\tau_{\mu\nu} A^\nu = \frac{\partial A^\tau}{\partial q^\nu} + \Gamma^\tau_{\mu\nu} A^\nu \tag{9}$$

In the second term on right are found the connection coefficients, here in form also called the Christoffel symbols of second kind. They originate from differentials of the metric tensor, in matter of fact they are a shorthand notation for an expression that shows how the covariant differentiation depends on the metric of the curved space. This is part of the Riemann theory of curved spaces, a pure mathematical theory that existed before Einstein started developing his theory of general relativity. As earlier, comma denotes partial derivation:

$$\Gamma^\tau_{\mu\nu} = \frac{1}{2} g^{\tau\kappa}\left(g_{\nu\kappa,\mu} + g_{\kappa\mu,\nu} - g_{\mu\nu,\kappa}\right) \tag{10}$$

The general formula for covariant derivative of a rank n covariant tensor, with indices $\mu_1, \mu_2, \ldots, \mu_n$:

$$A_{\mu_1\mu_2..\mu_n\,;\tau} = A_{\mu_1\mu_2..\mu_n\,,\tau} - \Gamma^{\sigma}_{\mu_1\tau} A_{\sigma\mu_2..\mu_n} - \Gamma^{\sigma}_{\mu_2\tau} A_{\mu_1\sigma..\mu_n} - ... - \Gamma^{\sigma}_{\mu_n\tau} A_{\mu_1\mu_2...\sigma} \tag{11}$$

Vectors are a special case of tensors, they are called rank 1 tensors. The metric tensor and the Ricci tensor with two indices are of rank 2, and the curvature tensor is of rank 4. The Christoffel symbols are not tensors (they do not transform like tensors), but consists of four rank 2 (lower indices) tensors, labeled by the upper index.

When we talk about tensors and vectors here it is in matter of fact fields. That is because the components of them are functions of the coordinates, so there is a an actual vector or tensor defined for every point in space. But note that for example the metric tensor is not a tensor field "in space". The tensor field that the metric tensor expresses is the field of space-time geodesic lines. It is thus the very framework of space-time, and it gives how space-time curves.

3. Curvature Tensors

Take first the covariant derivative of an arbitrary vector

$$a_{\mu\,;\,\nu} = a_{\mu\,,\,\nu} - \Gamma^{\sigma}_{\mu\nu}\, a_{\sigma} \tag{12}$$

the "commutation law" for second covariant derivation of a vector is (for a proof see **Appendix A** below):

$$a_{\mu\,;\nu\tau} - a_{\mu\,;\tau\nu} = R^{\sigma}_{\mu\nu\tau}\, a_{\sigma} \tag{13}$$

Here we have the Riemann - Christoffel curvature tensor

$$R^{\sigma}_{\mu\nu\tau} = \Gamma^{\sigma}_{\mu\tau,\nu} - \Gamma^{\sigma}_{\mu\nu,\tau} + \Gamma^{\rho}_{\mu\tau}\,\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\rho}_{\mu\nu}\,\Gamma^{\sigma}_{\rho\tau} \tag{14}$$

Because the number of dimensions is 4, it has $4^4 = 256$ components. However, it has a number of symmetries and identities, which makes that the number of independent components is no more than $N^2(N^2\ 1)/12 = 20$. From (14) is immediately seen that it is symmetrical in the last two indices:

$$R^{\sigma}_{\mu\nu\tau} = -R^{\sigma}_{\mu\tau\nu} \tag{15}$$

An other of the identities is the "cyclic identity", which is also easy to see

$$R^{\sigma}_{\mu\nu\tau} + R^{\sigma}_{\tau\nu\mu} + R^{\sigma}_{\nu\mu\tau} = 0 \tag{16}$$

Differentiating equation (13), rearranging the indices, and using the symmetry properties of the curvature tensor we get (see **Appendix B** below), [Foster - Nightingale ch.3. p. 77], [Bergmann Ch. X p. 168], [Vector Analysis, p. 206] :

$$a_{\mu\,;\nu\tau\eta} - a_{\mu\,;\,\nu\vartheta\tau} = R^{\sigma}_{\tau\mu\eta}\, a_{\sigma;\nu} + R^{\sigma}_{\tau\nu\eta}\, a_{\mu;\,\sigma} \tag{17}$$

Differentiate now further the "tensor commutator law" equation (17). Then by rearranging the indices and using the symmetry properties of the curvature tensor we get [Bergmann ch.X p. 172], [Möller p. 344] (Appendix C ...)

$$\left(R^{\sigma}_{\mu\nu\tau\,;\,\eta} + R^{\sigma}_{\mu\tau\eta\,;\nu} + R^{\sigma}_{\mu\eta\nu\,;\,\tau}\right) a_{\sigma} = 0 \tag{18}$$

Because the vector a_{σ} is not supposed to be zero all over we get the result known as the"Bianchi identities":

$$R^{\sigma}_{\mu\nu\tau\,;\,\eta} + R^{\sigma}_{\mu\tau\eta\,;\nu} + R^{\sigma}_{\mu\eta\nu\,;\,\tau} = 0 \tag{19}$$

The symmetrical rank 2 tensor called "Ricci tensor" is a contraction of the curvature tensor in first and third index [Möller p. 344]:

$$R_{\mu\tau} = R^{\nu}_{\mu\nu\tau} = \Gamma^{\nu}_{\mu\tau,\nu} - \Gamma^{\nu}_{\mu\nu,\tau} + \Gamma^{\vartheta}_{\mu\tau}\Gamma^{\nu}_{\vartheta\nu} - \Gamma^{\vartheta}_{\mu\nu}\Gamma^{\nu}_{\vartheta\tau} \tag{20}$$

further, the curvature scalar" is defined as the contraction of the Ricci tensor [Möller p. 345]

$$R = R_{\mu}{}^{\mu} = R^{\mu}{}_{\mu} = g^{\mu\tau} R_{\mu\tau} = g^{\mu\tau}\left(\Gamma^{\nu}_{\mu\tau,\nu} - \Gamma^{\nu}_{\mu\nu,\tau} + \Gamma^{\vartheta}_{\mu\tau}\Gamma^{\nu}_{\vartheta\nu} - \Gamma^{\vartheta}_{\mu\nu}\Gamma^{\nu}_{\vartheta\tau}\right) \tag{21}$$

4. The Cosmological Field Equation

Contraction of the Bianchi identities (19) with respect to σ and ν leads to (see **Appendix D** below)

$$\frac{D}{\partial q^{\nu}}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 0 \tag{22}$$

This equation expresses the vanishing of the tensor divergence (contraction of tensor gradient by the summing index ν) of a symmetrical tensor, the Einstein tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{23}$$

By integration from eq. (22) we get the "cosmological field equation" .

$$G^{\mu\nu} - \Lambda^{\mu\nu} = \kappa\, T^{\mu\nu} \tag{24}$$

Here the tensor $T^{\mu\nu}$ is the stress-energy tensor, and the constant factor can be shown to be $\kappa = 8\pi C_g$, where $C_g = 6.6726 \times 10^{-11}\ \mathrm{N\,m^2/kg^2}$ is the gravitation constant.

On the left side of (24) there is also an integration constant in form of a constant tensor $\Lambda^{\mu\nu}$.
Assuming certain, quite natural, symmetry conditions it can be written

$$\Lambda^{\mu\nu} = \Lambda\, g^{\mu\nu} \tag{25}$$

where Λ is what is called the "cosmological constant".

The cosmological constant is a kind of freedom (from point of view of general relativity) on the total energy of the universe. It is expected to represent a field and potential energy of still unknown origin ("black energy") that causes universe continually to expand. It does not follow from the logic of general relativity, but it can be included and does not change the rest of the reasoning, except the overall expansion. It is supposedly important when treating the universe in the largest scales, and as a total, but in local applications it can in most cases be leaved out.

Sometimes it is preferable to use an other form of the Cosmological Field Equation

$$R^{\mu\nu} = \kappa\left(T^{\mu\nu} - \frac{1}{2} T\, g^{\mu\nu}\right) \tag{26}$$

Where $T = T_{\mu}{}^{\mu}$, and we have not set out any cosmological constant.

This is basically all the mathematical theory of General Relativity; the rest is in the applications.

Appendices

Appendix A.

Starting from the first covariant derivative (12) the second covariant derivative becomes

$$a_{\mu;\nu\tau} = \frac{\partial a_{\mu;\nu}}{\partial q^{\tau}} - \Gamma^{\sigma}_{\mu\tau}\, a_{\sigma;\nu} - \Gamma^{\sigma}_{\nu\tau}\, a_{\mu;\sigma} \tag{A.1}$$

The first (differential) term here is

$$\frac{\partial a_{\mu;\nu}}{\partial q^{\tau}} = \frac{\partial\left(a_{\mu,\nu} - \Gamma^{\sigma}_{\mu\nu}\, a_{\sigma}\right)}{\partial q^{\tau}} = a_{\mu,\nu\tau} - \Gamma^{\sigma}_{\mu\nu,\tau}\, a_{\sigma} - \Gamma^{\sigma}_{\mu\nu}\, a_{\sigma,\tau} \tag{A.2}$$

The second term on right s

$$\Gamma^{\sigma}_{\mu\tau} a_{\sigma;\nu} = \Gamma^{\sigma}_{\mu\tau}\left(a_{\sigma,\nu} - \Gamma^{\vartheta}_{\sigma\nu}\, a_{\vartheta}\right) = \Gamma^{\sigma}_{\mu\tau} a_{\sigma,\nu} - \Gamma^{\sigma}_{\mu\tau}\Gamma^{\vartheta}_{\sigma\nu}\, a_{\vartheta} \tag{A.3}$$

The third term on right becomes

$$\Gamma^{\sigma}_{\nu\tau}\, a_{\mu;\sigma} = \Gamma^{\sigma}_{\nu\tau}\left(a_{\mu,\sigma} - \Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta}\right) = \Gamma^{\sigma}_{\nu\tau} a_{\mu,\sigma} - \Gamma^{\sigma}_{\nu\tau}\Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta} \tag{A.4}$$

Then it becomes

$$\begin{aligned} a_{\mu;\nu\tau} &= a_{\mu,\nu\tau} - \Gamma^{\sigma}_{\mu\nu,\tau}\, a_{\sigma} - \Gamma^{\sigma}_{\mu\nu}\, a_{\sigma,\tau} - \Gamma^{\sigma}_{\mu\tau}\left(a_{\sigma,\nu} - \Gamma^{\vartheta}_{\sigma\nu}\, a_{\vartheta}\right) - \Gamma^{\sigma}_{\nu\tau}\left(a_{\mu,\sigma} - \Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta}\right) = \ldots \\ \ldots &= a_{\mu,\nu\tau} - \Gamma^{\sigma}_{\mu\nu,\tau}\, a_{\sigma} - \Gamma^{\sigma}_{\mu\nu}\, a_{\sigma,\tau} - \Gamma^{\sigma}_{\mu\tau}\, a_{\sigma,\nu} + \Gamma^{\sigma}_{\mu\tau}\Gamma^{\vartheta}_{\sigma\nu}\, a_{\vartheta} - \Gamma^{\sigma}_{\nu\tau} a_{\mu,\sigma} + \Gamma^{\sigma}_{\nu\tau}\Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta} \end{aligned} \tag{A.5}$$

Switching the last two indices

$$a_{\mu;\nu\tau} = a_{\mu,\tau\nu} - \Gamma^{\sigma}_{\mu\tau,\nu}\, a_{\sigma} - \Gamma^{\sigma}_{\mu\tau}\, a_{\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu}\, a_{\sigma,\tau} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\vartheta}_{\sigma\tau}\, a_{\vartheta} - \Gamma^{\sigma}_{\tau\nu} a_{\mu,\sigma} + \Gamma^{\sigma}_{\tau\nu}\Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta} \tag{A.6}$$

The difference of the two is expressed

$$a_{\mu;\nu\tau} - a_{\mu;\nu\tau} = a_{\mu,\nu\tau} - a_{\mu,\tau\nu} + P_{\mu\nu\tau} + Q_{\mu\nu\tau} \tag{A.7}$$

where we have gathered together all the terms with the potential vector

$$P_{\mu\nu\tau} = -\Gamma^{\sigma}_{\mu\tau,\nu}\, a_{\sigma} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\vartheta}_{\sigma\tau}\, a_{\vartheta} + \Gamma^{\sigma}_{\tau\nu}\Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta} - \left(-\Gamma^{\sigma}_{\mu\tau,\nu}\, a_{\sigma} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\vartheta}_{\sigma\tau}\, a_{\vartheta} + \Gamma^{\sigma}_{\tau\nu}\Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta}\right) \tag{A.8}$$

Gathering for those with its first derivative of the vector

$$Q_{\mu\nu\tau} = -\Gamma^{\sigma}_{\mu\nu}\, a_{\sigma,\tau} - \Gamma^{\sigma}_{\mu\tau}\, a_{\sigma,\nu} - \Gamma^{\sigma}_{\nu\tau} a_{\mu,\sigma} - \left(-\Gamma^{\sigma}_{\mu\tau}\, a_{\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu}\, a_{\sigma,\tau} - \Gamma^{\sigma}_{\tau\nu} a_{\mu,\sigma}\right) = 0 \tag{A.9}$$

It follows from the symmetry of Christoffel symbols that this later term vanishes. The former term can be written, switching the summation indices, and rearranging:

$$\begin{aligned} P_{\mu\nu\tau} &= -\Gamma^{\sigma}_{\mu\nu,\tau}\, a_{\sigma} + \Gamma^{\sigma}_{\mu\tau}\Gamma^{\vartheta}_{\sigma\nu}\, a_{\vartheta} + \Gamma^{\sigma}_{\nu\tau}\Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta} + \Gamma^{\sigma}_{\mu\tau,\nu}\, a_{\sigma} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\vartheta}_{\sigma\tau}\, a_{\vartheta} - \Gamma^{\sigma}_{\tau\nu}\Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta} = \ldots \\ \ldots &= \Gamma^{\sigma}_{\mu\tau,\nu}\, a_{\sigma} - \Gamma^{\sigma}_{\mu\nu,\tau}\, a_{\sigma} + \Gamma^{\sigma}_{\mu\tau}\Gamma^{\vartheta}_{\sigma\nu}\, a_{\vartheta} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\vartheta}_{\sigma\tau}\, a_{\vartheta} + \left(\Gamma^{\sigma}_{\nu\tau}\Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta} - \Gamma^{\sigma}_{\tau\nu}\Gamma^{\vartheta}_{\mu\sigma}\, a_{\vartheta}\right) \end{aligned} \tag{A.10}$$

Further using the symmetry of the Christoffel symbols the last part in parenthesis disappears, leaving

$$P_{\mu\nu\tau} = \Gamma^{\sigma}_{\mu\tau,\nu}\, a_{\sigma} - \Gamma^{\sigma}_{\mu\nu,\tau}\, a_{\sigma} + \Gamma^{\sigma}_{\mu\tau}\Gamma^{\vartheta}_{\sigma\nu}\, a_{\vartheta} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\vartheta}_{\sigma\tau}\, a_{\vartheta} \tag{A.11}$$

and switching summation indices

$$P_{\mu\nu\tau} = \Gamma^{\sigma}_{\mu\tau,\nu}\, a_{\sigma} - \Gamma^{\sigma}_{\mu\nu,\tau}\, a_{\sigma} + \Gamma^{\vartheta}_{\mu\tau}\Gamma^{\sigma}_{\vartheta\nu}\, a_{\sigma} - \Gamma^{\vartheta}_{\mu\nu}\Gamma^{\sigma}_{\vartheta\tau}\, a_{\sigma} = \left(\Gamma^{\sigma}_{\mu\tau,\nu} - \Gamma^{\sigma}_{\mu\nu,\tau} + \Gamma^{\vartheta}_{\mu\tau}\Gamma^{\sigma}_{\vartheta\nu} - \Gamma^{\vartheta}_{\mu\nu}\Gamma^{\sigma}_{\vartheta\tau}\right) a_{\sigma} \qquad \text{(A.12)}$$

The expression inside parenthesis is the Riemann- Christoffel curvature tensor (14), and the original equation (1A.7) now becomes

$$a_{\mu;\nu\tau} - a_{\mu;\tau\nu} = a_{\mu,\nu\tau} - a_{\mu,\tau\nu} - R^{\sigma}_{\mu\nu\tau}\, a_{\sigma} \qquad \text{(A.13)}$$

The partial derivatives on the right side vanish for all analytic a_{μ}, because then the partial derivatives do not depend on the order of differentiation: $a_{\mu,\nu\tau} = a_{\mu,\tau\nu}$, which further leads to the commutation law (13) for second covariant derivation of a vector.

Appendix B.

In order to prove (17) we first deduce the commutation relation for a general rank 2 covariant tensor $A_{\mu\nu}$, and then set $a_{\mu;\nu} = A_{\mu\nu}$. Textbooks tend to tell that it can be readily shown, but do not show. Anyhow, it is a good exercise in tensor index manipulation. The trick is doing everything in the right order so that the partial derivatives fall out nicely. Note that the Christoffel symbols have partial derivatives but not covariant derivatives, because they are not tensors but rather a part of the covariant derivative.

So this kind of expression is not valid: $\cancel{\Gamma}^{\sigma}_{\mu\tau;\vartheta}$ but this is valid: $\Gamma^{\sigma}_{\mu\tau,\vartheta}$

The second covariant derivative can be written as follows, if we first write the first the the last of the covariant derivatives into a form of partial derivatives and products with Christoffel symbols
Note: below all indices go 0 , ... , 3 in contrary to common convention using greek lettes for such indices.

$$A_{jk;qr} = A_{jk;q;r} = A_{jk;q,r} - \Gamma^{s}_{jr} A_{sk;q} - \Gamma^{s}_{kr} A_{js;q} - \Gamma^{s}_{qr} A_{jk;s} \qquad \text{(B.1)}$$

This is valid because because the first covariant derivative of a tensor also is a tensor. The first covariant derivative becomes

$$A_{jk;q} = A_{jk,q} - \Gamma^{n}_{jq} A_{nk} - \Gamma^{n}_{kq} A_{jn} \qquad \text{(B.2)}$$

Its partial derivative can be written, using the chain rule for the products with the Christoffel symbols

$$A_{jk;q,r} = A_{jk,qr} - \Gamma^{n}_{jq,r} A_{nk} - \Gamma^{n}_{jq} A_{nk,r} - \Gamma^{n}_{kq,r} A_{jn} - \Gamma^{n}_{kq} A_{jn,r} \qquad \text{(B.3)}$$

Insert it to the original second covariant derivative

$$\begin{aligned} A_{jk;qr} &= \left(A_{jk,qr} - \Gamma^{n}_{jq,r} A_{nk} - \Gamma^{n}_{jq} A_{nk,r} - \Gamma^{n}_{kq,r} A_{jn} - \Gamma^{n}_{kq} A_{jn,r}\right) - \ldots \\ &\ldots - \Gamma^{s}_{jr} A_{sk;q} - \Gamma^{s}_{kr} A_{js;q} - \Gamma^{s}_{qr} A_{jk;s} \end{aligned} \qquad \text{(B.4)}$$

Changing indices in the first covariant derivative and inserting in above we get

$$\begin{aligned} A_{jk;qr} &= \left(A_{jk,qr} - \Gamma^{n}_{jq,r} A_{nk} - \Gamma^{n}_{jq} A_{nk,r} - \Gamma^{n}_{kq,r} A_{jn} - \Gamma^{n}_{kq} A_{jn,r}\right) - \ldots \\ &\ldots - \Gamma^{s}_{jr}\left(A_{sk,q} - \Gamma^{n}_{sq} A_{nk} - \Gamma^{n}_{kq} A_{sn}\right) - \Gamma^{s}_{kr}\left(A_{js,q} - \Gamma^{n}_{jq} A_{ns} - \Gamma^{n}_{sq} A_{jn}\right) - \ldots \\ &\ldots - \Gamma^{s}_{qr}\left(A_{jk,s} - \Gamma^{n}_{js} A_{nk} - \Gamma^{n}_{ks} A_{jn}\right) \end{aligned} \qquad \text{(B.5)}$$

We multiply in the Christoffel symbols, and then make a copy of the result by switching indices q and r . Subtract the later from the former, noticing that differences in the dummy indices (summing indices) n or s do not matter, ant the symmetry of the Christoffel symbols. Terms marked with square brackets below take out each other in the subtraction, each equal pair numbered:

$$A_{jk;qr} = A_{jk,qr} - \Gamma^n_{jq,r} A_{nk} - \underbrace{\left[\Gamma^n_{jq} A_{nk,r}\right]}_{[1]} - \Gamma^n_{kq,r} A_{jn} - \underbrace{\left[\Gamma^n_{kq} A_{jn,r}\right]}_{[2]} - \underbrace{\left[\Gamma^s_{jr} A_{sk,q}\right]}_{[3]} + \Gamma^s_{jr}\Gamma^n_{sq} A_{nk} + \ldots$$

$$\ldots + \underbrace{\left[\Gamma^s_{jr}\Gamma^n_{kq} A_{sn}\right]}_{[8]} - \underbrace{\left[\Gamma^s_{kr} A_{js,q}\right]}_{[4]} + \underbrace{\left[\Gamma^s_{kr}\Gamma^n_{jq} A_{ns}\right]}_{[9]} + \Gamma^s_{kr}\Gamma^n_{sq} A_{jn} - \underbrace{\left[\Gamma^s_{qr} A_{jk,s}\right]}_{[5]} + \underbrace{\left[\Gamma^s_{qr}\Gamma^n_{js} A_{nk}\right]}_{[6]} + \underbrace{\left[\Gamma^s_{qr}\Gamma^n_{ks} A_{jn}\right]}_{[7]} \qquad \text{(B.6)}$$

and indices switched

$$A_{jk;rq} = A_{jk,rq} - \Gamma^n_{jr,q} A_{nk} - \underbrace{\left[\Gamma^n_{jr} A_{nk,q}\right]}_{[3]} - \Gamma^n_{kr,q} A_{jn} - \underbrace{\left[\Gamma^n_{kr} A_{jn,q}\right]}_{[4]} - \underbrace{\left[\Gamma^s_{jq} A_{sk,r}\right]}_{[1]} + \Gamma^s_{jq}\Gamma^n_{sr} A_{nk} + \ldots$$

$$\ldots + \underbrace{\left[\Gamma^s_{jq}\Gamma^n_{kr} A_{sn}\right]}_{[9]} - \underbrace{\left[\Gamma^s_{kq} A_{js,r}\right]}_{[2]} + \underbrace{\left[\Gamma^s_{kq}\Gamma^n_{jr} A_{ns}\right]}_{[8]} + \Gamma^s_{kq}\Gamma^n_{sr} A_{jn} - \underbrace{\left[\Gamma^s_{rq} A_{jk,s}\right]}_{[5]} + \underbrace{\left[\Gamma^s_{rq}\Gamma^n_{js} A_{nk}\right]}_{[6]} + \underbrace{\left[\Gamma^s_{rq}\Gamma^n_{ks} A_{jn}\right]}_{[7]} \qquad \text{(B.7)}$$

Now, what we have left is only:

$$A_{jk;qr} - A_{jk;rq} = A_{jk,qr} - \Gamma^n_{jq,r} A_{nk} - \Gamma^n_{kq,r} A_{jn} + \Gamma^s_{jr}\Gamma^n_{sq} A_{nk} + \Gamma^s_{kr}\Gamma^n_{sq} A_{jn} - \ldots$$

$$\ldots - A_{jk,rq} + \Gamma^n_{jr,q} A_{nk} + \Gamma^n_{kr,q} A_{jn} - \Gamma^s_{jq}\Gamma^n_{sr} A_{nk} - \Gamma^s_{kq}\Gamma^n_{sr} A_{jn} \qquad \text{(B.8)}$$

Then using the analyticity condition for the tensor components the second partial derivatives take out each other, and gathering the common terms:

$$A_{jk;qr} - A_{jk;rq} = \left(\Gamma^n_{jr,q} - \Gamma^n_{jq,r} + \Gamma^s_{jr}\Gamma^n_{sq} - \Gamma^s_{jq}\Gamma^n_{sr}\right) A_{nk} + \ldots$$

$$+ \left(\Gamma^n_{kr,q} - \Gamma^n_{kq,r} + \Gamma^s_{kr}\Gamma^n_{sq} - \Gamma^s_{kq}\Gamma^n_{sr}\right) A_{jn} + \ldots \qquad \text{(B.9)}$$

From the definition (14) of Riemann - Christoffel curvature tensor follows:

$$A_{jk;qr} - A_{jk;rq} = -R^n_{jqr} A_{nk} - R^n_{kqr} A_{jn} \qquad \text{(B.10)}$$

which leads to (17).

Appendix C.

(... is under construction) Differentiate the "tensor commutator law" equation (17)

$$a_{\mu;\nu\tau\eta} - a_{\mu;\nu\vartheta\tau} = R^{\sigma}_{\tau\mu\eta}\, a_{\sigma;\nu} + R^{\sigma}_{\tau\nu\eta}\, a_{\mu;\sigma}$$

... then by rearranging the indices and using the symmetry properties of the curvature tensor we get

$$\left(R^{\sigma}_{\mu\nu\tau;\eta} + R^{\sigma}_{\mu\tau\eta;\nu} + R^{\sigma}_{\mu\eta\nu;\tau}\right) a_{\sigma} = 0$$

Appendix D.

Contraction of the Bianchi identities (19) with respect to σ and ν leads to [Möller p. 345]

$$R^{\nu}_{.\mu\nu\tau;\eta} + R^{\nu}_{.\mu\tau\eta;\nu} + R^{\nu}_{.\mu\eta\nu;\tau} = 0 \qquad \text{(D.1)}$$

$$R_{\mu\tau;\eta} + R^{\nu}_{.\mu\tau\eta;\nu} - R_{\mu\eta;\tau} = 0 \qquad \text{(D.2)}$$

$$g^{\sigma\mu}\left(R_{\mu\,\tau\,;\,\eta}+R^{\nu}_{.\,\mu\tau\eta;\nu}-R_{\mu\eta\,;\,\tau}\right)=0 \tag{D.3}$$

$$R^{\sigma}_{\tau\,;\,\eta}+R^{\nu\sigma}_{\,.\,.\,\tau\eta;\nu}-R^{\sigma}_{\eta\,;\,\tau}=0 \tag{D.4}$$

$$R^{\sigma}_{\tau\,;\,\eta}-R^{\sigma\nu}_{\,.\,.\,\tau\,\eta;\nu}-R^{\sigma}_{\eta\,;\,\tau}=0 \tag{D.5}$$

Further contraction with indices σ and τ gives

$$R^{\tau}_{\tau\,;\,\eta}-R^{\tau\nu}_{\,.\,.\,\tau\,\eta;\nu}-R^{\tau}_{\eta\,;\,\tau}=0 \tag{D.6}$$

$$R_{\,;\,\eta}-R^{\nu}_{\eta;\nu}-R^{\tau}_{\eta\,;\,\tau}=0 \tag{D.7}$$

$$R_{\,;\,\eta}-2R^{\nu}_{\eta\,;\nu}=0 \tag{D.8}$$

$$g^{\mu\eta}\left(R_{\,;\,\eta}-2R^{\nu}_{\eta\,;\nu}\right)=0 \tag{D.9}$$

$$g^{\mu\eta}R_{\,;\,\eta}-2g^{\mu\eta}R^{\nu}_{\eta\,;\nu}=0 \tag{D.10}$$

Because covariant differentiation of the metric tensor generally yields zero:

$$\frac{1}{2}g^{\mu\eta}R_{;\eta}-R^{\mu\nu}{}_{;\nu}=0 \tag{D.11}$$

$$\frac{1}{2}g^{\mu\nu}R_{;\nu}-R^{\mu\nu}{}_{;\nu}=0 \tag{D.12}$$

$$\frac{D}{\partial q^{\nu}}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)=0 \tag{D.13}$$

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