Schwarzschild Metric

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Line Element and Coordinates

The conditions of the Schwarzschild solution in General Relativity are commonly given as follows: in a locally inertial system of coordinates the space-time is

[Foster, Nightingale ch. 4.5]

(i) static, that is, homogeneous in the time coordinate,
(ii) spherically symmetric in three dimensions,
(iii) empty, except possibly in one point,
(iv) asymptotically Euclidean long away from any mass point.

It has been shown that the general static spherically symmetrical solution has a line element of type :

$$(c\ d\tau)^2 = A(r)\ (c\ dt)^2 - \left(B(r)\ dr^2 + r^2\ d\theta^2 + r^2\ \sin^2\theta\ d\phi^2\right) \tag{1}$$

The condition (iv) will be true if both

$$A(r) \to 1 \text{ , and } B(r) \to 1 \text{ when } r \to \infty\ . \tag{2}$$

In the flat space case the three-space Cartesian coordinates expressed using spherical coordinates r, θ , ϕ ; time t is the fourth one:

$$\begin{cases} q_0 = c\ t \\ q_1 = x = r\ \sin\theta\ \cos\phi \\ q_2 = y = r\ \sin\theta\ \sin\phi \\ q_3 = z = r\ \cos\theta \end{cases} \tag{3}$$

The curvilinear covariant coordinates are the 4-dimensional cylinder coordinates, correspondingly expressed with the Cartesion coordinates of a flat space-time:

$$\begin{cases} q^0 = c\ t \\ q^1 = r = \sqrt{x^2 + y^2 + z^2} \\ q^2 = \theta = \operatorname{acos}\left(\dfrac{z}{r}\right) \\ q^3 = \phi = \operatorname{atan}\left(\dfrac{y}{x}\right) \end{cases} \tag{4}$$

Metric Tensor and Metric Connection

The line-element (1) gives us the elements of the metric tensor.

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \tag{6}$$

The metric connection is given by the Christoffel symbols, of first respective second kind

$$\Gamma_{\tau\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\nu\tau} + \partial_\nu g_{\tau\mu} - \partial_\tau g_{\mu\nu}\right) \tag{7}$$

$$\Gamma^{\sigma}_{\mu\nu} = g^{\sigma\tau}\,\Gamma_{\tau\mu\nu} \tag{8}$$

Following alternatives give all of the nonzero components, when the metric tensor is diagonal:

$$\begin{cases} \nu = \mu = \tau & \Gamma_{\mu\mu\mu} = \frac{1}{2}\left(\partial_\mu g_{\mu\mu} + \partial_\mu g_{\mu\mu} - \partial_\mu g_{\mu\mu}\right) = \frac{1}{2}\ \partial_\mu g_{\mu\mu} \\ \nu \neq \mu = \tau & \Gamma_{\mu\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\nu\mu} + \partial_\nu g_{\mu\mu} - \partial_\mu g_{\mu\nu}\right) = \frac{1}{2}\ \partial_\nu g_{\mu\mu} \\ \nu = \mu \neq \tau & \Gamma_{\tau\mu\mu} = \frac{1}{2}\left(\partial_\mu g_{\mu\tau} + \partial_\mu g_{\tau\mu} - \partial_\tau g_{\mu\mu}\right) = -\frac{1}{2}\ \partial_\tau g_{\mu\mu} \\ \nu = \tau \neq \mu & \Gamma_{\nu\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\nu\nu} + \partial_\nu g_{\nu\mu} - \partial_\nu g_{\nu\mu}\right) = \frac{1}{2}\ \partial_\mu g_{\nu\nu} \end{cases} \tag{9}$$

And we also have

$$g^{\sigma\sigma} = \frac{1}{g_{\sigma\sigma}} \tag{10}$$

Calculation of the Christoffel Symbols

Nonzero components with upper index 0 :

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{g_{00}}\Gamma_{001} = \frac{1}{g_{00}}\frac{1}{2}\partial_1 g_{00} = \frac{1}{A}\frac{1}{2}\frac{\partial}{\partial r}A(r) = \frac{A'}{2A} \tag{11}$$

Nonzero components with upper index 1:

$$\Gamma^1_{00} = \frac{1}{g_{11}}\Gamma_{100} = \frac{1}{g_{11}}\left(-\frac{1}{2}\partial_1 g_{00}\right) = \frac{1}{-B}\left(-\frac{1}{2}\frac{\partial}{\partial r}A(r)\right) = \frac{A'}{2B} \tag{12}$$

$$\Gamma^1_{11} = \frac{1}{g_{11}}\Gamma_{111} = \frac{1}{g_{11}}\frac{1}{2}\partial_1 g_{11} = \frac{1}{-B}\frac{1}{2}\frac{\partial}{\partial r}\left(-B(r)\right) = \frac{B'}{2B} \tag{13}$$

$$\Gamma^1_{22} = \frac{1}{g_{11}}\Gamma_{122} = \frac{1}{g_{11}}\left(-\frac{1}{2}\partial_1 g_{22}\right) = \frac{1}{-B}\left(-\frac{1}{2}\partial_1\left(-r^2\right)\right) = -\frac{r}{B} \tag{14}$$

$$\Gamma^1_{33} = \frac{1}{g_{11}}\Gamma_{133} = \frac{1}{g_{11}}\left(-\frac{1}{2}\partial_1 g_{33}\right) = \frac{1}{-B}\left(-\frac{1}{2}\partial_1\left(-r^2\sin^2\theta\right)\right) = -\frac{r}{B}\sin^2\theta \tag{15}$$

Nonzero components with upper index 2:

$$\Gamma^2_{33} = \frac{1}{g_{22}}\Gamma_{233} = \frac{1}{g_{22}}\left(-\frac{1}{2}\partial_2 g_{33}\right) = \frac{1}{-r^2}\left[-\frac{1}{2}\frac{\partial}{\partial\theta}\left(-r^2\sin^2\theta\right)\right] = -\sin\theta\cos\theta \tag{16}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{g_{22}}\Gamma_{212} = \frac{1}{g_{22}}\frac{1}{2}\partial_1 g_{22} = \frac{1}{-r^2}\frac{1}{2}\frac{\partial}{\partial r}\left(-r^2\right) = \frac{-r}{-r^2} = \frac{1}{r} \tag{17}$$

Nonzero components with upper index 3 :

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{g_{33}}\Gamma_{313} = \frac{1}{g_{33}}\frac{1}{2}\partial_1 g_{33} = \frac{1}{-r^2\sin^2\theta}\frac{\partial}{\partial r}\left(-r^2\sin^2\theta\right) = \frac{-r\sin^2\theta}{-r^2\sin^2\theta} = \frac{1}{r} \tag{18}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{\Gamma_{323}}{g_{33}} = \frac{1}{2}\frac{\partial_2 g_{33}}{g_{33}} = \frac{1}{2}\frac{\frac{\partial}{\partial\theta}\left(-r^2\sin^2\theta\right)}{-r^2\sin^2\theta} = \frac{-r^2\sin\theta\cos\theta}{-r^2\sin^2\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta \tag{19}$$

Field Equation for the Schwarzchild Line-element

The Ricci curvarure tensor is defined (ref. Gravitation p.224):

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{20}$$

The diagonal elements of the Ricci-tensor become (the rest are identically zero):

$$\left\{\begin{aligned}
R_{00} &= \frac{A''}{2\,B} - \frac{A'}{4\,B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{r\,B}\\
R_{11} &= -\frac{A''}{2\,A} + \frac{A'}{4\,A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{r\,B}\\
R_{22} &= 1 + \frac{1}{B} - \frac{r}{2\,B}\left(\frac{A'}{A} - \frac{B'}{B}\right)\\
R_{33} &= \left[1 + \frac{1}{B} - \frac{r}{2\,B}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right]\sin(\theta)^2
\end{aligned}\right. \tag{21}$$

further, we get the curvature scalar as the contraction of the Ricci tensor

$$R = g^{\mu\nu}\,R_{\mu\nu} \tag{22}$$

In the diagonal case and expressed (summation over μ):

$$R = \frac{1}{g_{\mu\mu}} R_{\mu\mu} \tag{23}$$

The curvature scalar becomes

$$R = \frac{A''}{A\ B} - \frac{A'}{2\ A\ B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{2}{r^2}\left[\frac{1}{B} - 1 + \frac{r}{B}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] \tag{24}$$

The Einstein-tensor is defined:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\ g_{\mu\nu} \tag{25}$$

Also the Einstein-tensor is here diagonal, and the diagonal elements become

$$G_{00} = \left(\frac{1}{r^2} - \frac{1}{B\ r^2} + \frac{B'}{B^2\ r}\right) A \tag{26}$$

$$G_{11} = -\left(\frac{1}{r^2} - \frac{1}{B\ r^2} - \frac{A'}{A\ B\ r}\right) B \tag{27}$$

$$G_{22} = -\left[-\frac{A''}{2\ A\ B} + \frac{A'}{4\ A\ B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{2\ rB}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] r^2 \tag{28}$$

$$G_{33} = -\left[-\frac{A''}{2\ A\ B} + \frac{A'}{4\ A\ B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{2\ rB}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] r^2\ \sin(\theta)^2 \tag{29}$$

Contravariant form

$$G^{00} = \left(g^{00}\right)^2 G_{00} = \left(\frac{1}{r^2} - \frac{1}{B\ r^2} + \frac{B'}{B^2\ r}\right) \frac{1}{A} \tag{30}$$

$$G^{11} = \left(g^{11}\right)^2 G_{11} = -\left(\frac{1}{r^2} - \frac{1}{B\ r^2} - \frac{A'}{A\ B\ r}\right) \frac{1}{B} \tag{31}$$

$$G^{22} = \left(g^{22}\right)^2 G_{22} = -\left[-\frac{A''}{2\ A\ B} + \frac{A'}{4\ A\ B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{2\ rB}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] \frac{1}{r^2} \tag{32}$$

$$G^{33} = \left(g^{33}\right)^2 G_{33} = -\left[-\frac{A''}{2\ A\ B} + \frac{A'}{4\ A\ B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{2\ rB}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] \frac{1}{r^2\ \sin(\theta)^2} \quad (32)$$

The Cosmological field equation:

$$G^{\mu\nu} = \kappa\ T^{\mu\nu} + \Lambda\ g^{\mu\nu} \tag{33}$$

The constant tensor on right hand side is the stress-energy tensor. The scalar value $\kappa = 8\,\pi\,G$.

The emptiness condition (iii) can be expressed using the field equation (33), and assuming that the components of the stress-energy tensor are identically zero. One then gets three independent equations and can solve out the functions $A(r)$ and $B(r)$.

$$\begin{cases} \dfrac{1}{B\,r^2} - \dfrac{1}{r^2} - \dfrac{B'}{B^2\,r} = 0 & (34) \\ \dfrac{1}{B\,r^2} - \dfrac{1}{r^2} + \dfrac{A'}{A\,B\,r} = 0 & (35) \\ \dfrac{A''}{2\,A\,B} + \dfrac{A'}{4\,A\,B}\left(\dfrac{A'}{A} + \dfrac{B'}{B}\right) - \dfrac{1}{2\,rB}\left(\dfrac{A'}{A} - \dfrac{B'}{B}\right) = 0 & (36) \end{cases}$$

These become further

$$\begin{cases} A\,B - AB^2 - B'\,A\,r = 0 & (37) \\ -AB + AB^2 - A'\,B\,r = 0 & (38) \\ r\,\dfrac{A''}{A} + \dfrac{r}{2}\,\dfrac{A'}{A}\left(\dfrac{A'}{A} + \dfrac{B'}{B}\right) - \left(\dfrac{A'}{A} - \dfrac{B'}{B}\right) = 0 & (39) \end{cases}$$

The first two equations by addition give a condition for A and B

$$A'\,B + B'\,A = 0 \tag{40}$$

Then

$$\frac{d}{dr}(A\,B) = 0 \tag{41}$$

That means $A\,B$ is constant, and from the limit condition (2) follows:

$$A\,B = 1 \tag{42}$$

Because now

$$B = \frac{1}{A} \qquad \Rightarrow \qquad B' = \frac{-A'}{A^2} \tag{43}$$

Subtituting the results (40) and (41) into the third equation (39) gives

$$r\,A'' + 2\,A' = 0 \tag{44}$$

This is a linear differential equation of first degree for the first differential of A, with has the solution

$$A'(r) = \frac{K}{r^2} \tag{45}$$

Integrating this further we get

$$A(r) = \frac{-K}{r} + C \tag{46}$$

Here K and C are integration constants. From conditions (2) follows that C = 1, so we get

$$A(r) = 1 - \frac{K}{r} \tag{47}$$

We can now write down the line-element of the Schwarzschild metrics

$$(c\ d\tau)^2 = \left(1 - \frac{K}{r}\right) dt^2 - \left(1 - \frac{K}{r}\right)^{-1} dr^2 - r^2\ d\,\Omega^2 \tag{48}$$

where the angle Ω is defined so that

$$d\,\Omega^2 = d\theta^2 + \sin^2\theta\ d\phi^2 \tag{49}$$

The Ricci tensor and the stress-energy tensor now vanish, as was presupposed, exept possibly in the singular point where $r = 0$.

We now have

$$A = 1 - \frac{K}{r} \qquad B := \frac{1}{A} \qquad B' = \frac{-A'}{A^2} \qquad A' = \frac{K}{r^2} \tag{50}$$

The non-zero connection components (Christoffel symbols of type 2) become now

$$\begin{cases} \Gamma^0_{01} = \dfrac{A'}{2A} = \dfrac{K}{2r(r-K)} \\ \Gamma^1_{00} = \dfrac{A'}{2B} = \dfrac{A'A}{2} = \dfrac{K(r-K)}{2r^3} \\ \Gamma^1_{11} = \dfrac{B'}{2B} = -\dfrac{A'A}{2A^2} = -\dfrac{A'}{2A} = \dfrac{-K}{2r(r-K)} \\ \Gamma^1_{22} = -\dfrac{r}{B} = -rA = -(r-K) \\ \Gamma^1_{33} = -\dfrac{r}{B}\sin^2\theta = -rA\sin^2\theta = -(r-K)\sin^2\theta \end{cases} \tag{51}$$

The rest are not affected.

Classical Gravitation Field and Potential

We have already investigated the space structure in the special case of on point-like gravitation mass, which approximation in most cases is good enough for investigating of motion in the vicinity of the mass. We now have the metric tensor component

$$g_{00} = 1 - \frac{K}{r} \tag{52}$$

From the condition (iv) in follow at the Newtonian limit:

$$h_{00} = \frac{-K}{r} = \frac{-2\ V}{m\ c^2} \qquad (53)$$

Here m is a constant, a Newtonian small "mass-probe" with a negligble own gravitation, V is the classical gravitational potential, G the gravitation constant, and M the gravitating mass.

$$V = \frac{-G\ M\ m}{r} \qquad (54)$$

We see then, that

$$K = \frac{2\ G\ M}{c^2} \qquad (55)$$

Black Holes and Singularities

In the the metric tensor (46) , the first diagonal component has a mathematical singularity at $r = 0$. The second diagonal component of the metric tensor has a coordinate singularity at: $r = K$. This later singularity is dependent of the choice of the coordinates, and can be circumvent. It appears at the distance known as the "Schwarzschild radius". At this distance the "escape velocity" is c and below it > c , so nothig, not even light, can escape from inside this limit. Denote this radius with r_0 .

$$r_0 = \frac{2\ G\ M}{c^2} \qquad (56)$$

In order to investigate how the solution behaves near the singular radius, we take a "time slice" of the four-dimensional space-time. That means, $t =$ constant and $dt = 0$. We assume causal connection and velocities less than c, so it is a time-like slice and the whole should become negative.

We denote a positive distance element dR^2, then from line-element (48) we get

$$(c\ d\tau)^2 = -dR^2 = -\left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2\ d\,\Omega^2 \qquad (57)$$

Now the line-element here (57) is three-dimensional, but the space it describes is not a classical 3-space, it is still curved. On a surface of a sphere in this curved 3-space $r =$ constant and $dR = r\ d\Omega$, which leads to an Euclidean-look-alike arch-length expression. The space thus appears to be curved only in the radial direction, and equation (57) expresses the curving of the space in that direction. It means, we can take $\Omega =$ constant, and so $d\Omega = 0$.

Now we can solve out the radial distance along the radial geodesic line.

$$dR^2 = \frac{dr^2}{\left(1 - \frac{r_0}{r}\right)} \qquad (58)$$

and we get the 3-dimensional radial distance in the space-time slice

$$dR = \frac{dr}{\sqrt{1 - \frac{r_0}{r}}} \tag{59}$$

Then by integration we get a simple expression; a is an integration constant.

$$R(r) = \sqrt{r\left(r - r_0\right)} - r_0 \ln\left[\frac{1}{\sqrt{r_0}}\left(\sqrt{r} + \sqrt{r - r_0}\right)\right] + a \tag{60}$$

Setting here $R\left(r_0\right) = 0$ gives $r_0 \ln\left(\sqrt{-1}\right) + a = 0$ (61)

and we get for the constant a an imaginary value, which hardly is feasible here.
Set in stead $r = r_0$, then

$$R\left(r_0\right) = \left(r_0\right) \ln\left(\frac{1}{\sqrt{r_0}}\sqrt{r_0}\right) + a = a \tag{62}$$

On the Newtonian limit we should have R smoothly nearing r, when r_0 goes to zero, clearly suggesting that we should set $a = 0$ so that $R\left(r_0\right) = 0$. The function (60) now becomes

$$R(r) = r_0\left[\sqrt{\frac{r}{r_0}\left(\frac{r}{r_0} - 1\right)} - r_0 \ln\left(\sqrt{\frac{r}{r_0}} + \sqrt{\frac{r}{r_0} - 1}\right)\right] \tag{63}$$

This is clearly real-valued when $r > r_0$. For $r = r_0$ we get $R = 0$, as was assumed.
Now if $r < r_0$ then we must have $r < 0$ in order to get real values from the first term. The second term is then a logarithm of an imaginary number, which is not unabigiously defined. The conclusion is that we have meaningful physical paths only outside the Schwartzschild radius. This is anyway the conclusion from the presented approach. Other approaches are possible giving different conclusions.

It is now, in the current approach, the geodesic distance R, measured from the Schwarzschild radius r_0, that is the physical radial distance, and the coordinate r is rather to be seen as a formal mathematical parameter. The later can, though, be interpreted as a physical lengt: it is the circumference of a circle round the mass center per 2π, thus literally "going round" the the black hole. It is practical to use this formal mathematical quantity as the "r-coordinate" in calculations. And long away from the Schwartzschild radius it also has nearly the same value as the physical geodesic distance.

Is there a physical singularity? Note that the the word "singularity" is originally not physics, it is a mathematical term. In physics the use is even more symbolic: assuming certain simplifications in the physical case one ends up an equation that has a mathematical singularity at $r = 0$. There is no mystic in it. Real physical situations are notoriously difficult to solve, so approximate simplifications are frequantly used in stead. The resuts are then not exactly correct but are expected to be near enough for any astrophysical purposes. Moreover, the result is proven to be correct for any limited mass distribution. But the universe is not limited, and so real mass distributions are not limited. We can then really not tell if there is any factual singularity, it is merely a label of the mathematical characteristic of the approximate solution.

Photon Sphere

One more spectacular distance from a point-mass (not necessarily a "black hole") is the "photon sphere". It is where photons go to orbit. (The theory of General Relativity does not include quantum particles like photons, but we can assume the existence of a tiny object with exactly the velocity of light, and call it a "photon"). We start from the line-element

$$c^2\, d\tau^2 = \left(1 - \frac{r_0}{r}\right) c^2\, dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \tag{64}$$

We search a circular orbit that is a"null"-geodesic:

$$c^2\, d\tau^2 = 0 \tag{65}$$

The radial velocity, and other angular velocity $d\phi$ is taken to be zero. The we get

$$0 = \left(1 - \frac{r_0}{r}\right) c^2\, dt^2 - r^2\, d\theta^2 \tag{66}$$

Then we get

$$\frac{d\phi}{dt} = \frac{c\sqrt{r^2 - r_0\, r}}{r^2} = \frac{c}{r}\sqrt{1 - \frac{r_0}{r}} \tag{67}$$

Geodesic equation:

$$\frac{d^2 q^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu} \frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau} = 0 \tag{68}$$

A condition for the circular geodesic can be given using the geodesic equation corresponding the coordinate r :

$$\frac{d^2 r}{d\tau^2} + \Gamma^1_{\mu\nu} \frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau} = 0 \tag{69}$$

On the circular orbit $\frac{dr}{d\tau} = 0$ then

$$0 = \Gamma^1_{00}\left(\frac{d(ct)}{d\tau}\right)^2 + \Gamma^1_{11}\left(\frac{dr}{d\tau}\right)^2 + \Gamma^1_{22}\left(\frac{d\theta}{d\tau}\right)^2 + \Gamma^1_{33}\left(\frac{d\phi}{d\tau}\right)^2 \tag{70}$$

further assuming $d\phi = 0$ gives

$$\frac{r_0\,(r - r_0)}{2\, r^3}\left(\frac{c\, dt}{d\tau}\right)^2 - (r - r_0)\left(\frac{d\phi}{d\tau}\right)^2 = 0 \tag{71}$$

and further

$$\frac{r_0}{2\, r^3}\left(\frac{c\, dt}{d\tau}\right)^2 = \left(\frac{d\phi}{d\tau}\right)^2 \tag{72}$$

and we get
$$\frac{r_0}{2\ r^3}\ c^2 = \left(\frac{d\phi}{dt}\right)^2 \tag{73}$$

From earlier we have the corresponding expression, and making them equal gives

$$\frac{c^2}{r^2}\left(1 - \frac{r_0}{r}\right) = \frac{r_0}{2\ r^3}\ c^2 \tag{74}$$

which gives for the photon orbit radius

$$r_\gamma = \frac{3}{2}\ r_0 \tag{75}$$

Althoug it seems clear that photons go to orbit at this distance, it is not a stable orbit. Wery few photons might stay there any substantial time.

General particle motion in vicinity of a heavy mass

The geodesic equation (Euler-Lagrange eq.) is:

$$\frac{d}{d\tau}\left(\frac{d}{du^\mu}L\right) - \frac{d}{dq^\mu}L = 0 \tag{76}$$

$$u^\mu = \frac{dq^\mu}{d\tau} \tag{77}$$

Where we get the lagrangian from the line-element (1):

$$L\left(u^\sigma, q^\sigma\right) = \frac{1}{2} g_{\mu\nu}\ u^\mu\ u^\nu = \frac{1}{2}\left[A(r)\ c^2\left(u^t\right)^2 - B(r)\left(u^r\right)^2 - r^2\left(u^\theta\right)^2 - r^2\ \sin^2\theta\left(u^\phi\right)^2\right] \tag{78}$$

Because of symmetry there is no loss of generality if we assume that the particle is moving on an "equatorial" plane. Assume $\theta = \pi/2$, so that the velocity corresponding θ is zero, and $\sin\theta = 1$. [Foster & Nightingale, p. 106]

$$L\left(u^\sigma, q^\sigma\right) = \frac{1}{2} g_{\mu\nu}\ u^\mu\ u^\nu = \frac{1}{2}\left[A(r)\ c^2\left(u^t\right)^2 - B(r)\left(u^r\right)^2 - r^2\left(u^\phi\right)^2\right] \tag{79}$$

We need only to calculate the second geodesic equation (μ= 1) :

$$\frac{d}{d\tau}\left(\frac{d}{du^\mu}L\right) = \frac{d}{d\tau}\left(-B(r)\ u^r\right) = -B(r)\ a^r \tag{80}$$

$$a^r = \frac{d^2\ q^\mu}{d\tau^2} \tag{81}$$

$$\frac{d}{dq^r} L = \frac{A'(r)}{2} c^2 \left(u^t\right)^2 - \frac{B'(r)}{2} \left(u^r\right)^2 - r \left(u^\phi\right)^2 \tag{82}$$

It reduces to

$$-B(r)\ a^r - \frac{A'(r)}{2} c^2 \left(u^t\right)^2 + \frac{B'(r)}{2} \left(u^r\right)^2 + r \left(u^\phi\right)^2 = 0 \tag{83}$$

Now t and ϕ are cyclic (preserved) coordinates:

$$\frac{d}{du^t} L = A(r)\ c^2\ u^t = k \tag{84}$$

$$\frac{d}{du^\phi} L = r^2\ c^2\ u^\phi = h \tag{85}$$

where k and h are constants. The later equation clearly expresses the conservation of angular momentum. The former equation is analogous to that, but expressing in stead conservation of something in time, which means conservation of energy.

We get from (84) between the coordinate time t and the proper time τ :

$$u^t = \frac{dt}{d\tau} = \frac{k}{c^2\ A(r)} \tag{86}$$

and so

$$t = \frac{k\ \tau}{c^2\ A(r)} \tag{87}$$

We also get from the line element, because $d\tau/d\tau = 1$:

$$c^2 = A(r)\ c^2 \left(u^t\right)^2 - B(r) \left(u^r\right)^2 - r^2 \left(u^\phi\right)^2 \tag{88}$$

... sorry .... this part is not ready made yet ...