

## 2. The Quantum Harmonic Oscillator

By: Seppo Nurmi

### 2.1 The Fall of Classical Physics

We have in the former chapter discussed the classical theory of the harmonic oscillator. The results from the theory are in good agreement with measured facts as long as the oscillator is macroscopic, that is, of substantial size. When the oscillator is of atomic size, as it is in the case of the diatomic molecule, the agreement with experiments is not good at all. For example we can in the classical theory have any energy, and also very small oscillations where energy is nearly zero, and the oscillator can entirely stand still in zero oscillation energy. Oscillators of molecular size do not at all follow these classical results as was later realized. The lowest possible oscillation energy is in fact greater than zero, and all the possible oscillation energies over that lowest "zero-point energy" come in discrete steps.

Even more serious was following problem, which more or less initiated the quantum era: any amount of energy, even small amounts, can in classical physics go with all frequencies, also very high frequencies, and as a complication to that, when the oscillation frequency increases then the energy increases even more. Because there are tremendously many small molecular oscillators in the world, this would lead to that most of the energy available in the world would at end be absorbed by the molecular oscillators in very high frequencies. This situation was catastrophic, not for the world, such energy absorption into high frequencies was not observed at all, but it was catastrophic for the classical physics because the prediction was terribly wrong. It was called "the ultraviolet catastrophe" because the frequency of ultraviolet light was the highest known at that time (in about year 1900).

In reality it was found that energy that goes to higher frequencies increases first rapidly, but reaches soon a maximum and then gradually decreases again. The maximum of the curve, where it is located in the energy scale and its height, depends on the temperature. This is called the "black body" radiation curve. An "absolutely black body" is a non-colored idealized object, an aid for theoretical calculations, but real objects at large behave very similarly. Investigation of this radiation curve led Max Planck (German physicist, 1858 - 1947) to suggest a solution in form of a quantization rule. Planck's rule was first assumed only as tentative, it was based on fitting theoretical curve forms on experimental results. It says: energy  $E$  of an oscillator comes in bunches bound to the frequency  $\nu$  by a constant  $h$

$$E = h \nu \tag{2.1}$$

This Planck's constant is very small  $h = 6,6256 \cdot 10^{-34}$  Js , which explains why energy in macroscopic oscillators appear to be (nearly) continuous. It is necessary for energy to reach the amount  $E=h\nu$  in order to be absorbed (or emitted).

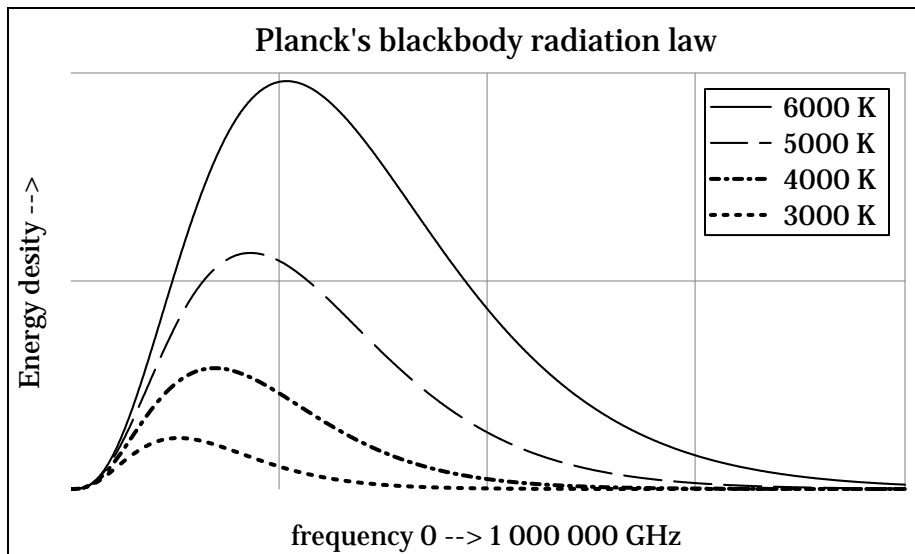
Because this amount, even if it is very small for low frequencies, becomes substantially high for higher frequencies, it makes it easier for energy go in lower frequencies, and exceedingly difficult for higher energies. Consequently the absorption rate goes first up and then down again after a certain maximum. The "catastrophe" is thus avoided. What is said of absorption also holds other way round, for emission of energy: big chunks of energy are not so easy to get rid off, and very small amounts do not give off much effect, so the maximum energy emission rate is somewhere mid between. By making the assumption (2.1) and substituting it to already known formulas of energy density in statistical arrays of oscillators, Planck found (he was an expert of statistical physics) the formula now called the "Planck's radiation formula":

$$E_{\text{density}} = \frac{8 \pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} \quad (2.2)$$

It gives the energy density per unit volume and frequency for a black body object of surface temperature T in degrees Kelvin (K, degrees from absolute zero -273.16 °C). We have here the constants:

velocity of light  $c = 2.998 \times 10^8 \frac{\text{m}}{\text{s}}$

and Boltzmann constant  $k = 1.3805 \times 10^{-23} \frac{\text{J}}{\text{K}}$



The plot above shows energy density as it varies with increasing frequency. The curves are for different blackbody temperatures from 3000 K to 6000 K. They are also called the "blackbody radiation curves" for the given temperatures. (It is not unusual drawing them with waveleght as argument, which turns them left to right.)

The formula (2.2) has found to be correct, although it was originally based on semi-classical and not even entirely correct assumptions. This Planck energy curve is continuous, although strictly bound to the frequency. According to Planck's rule energy is quantized in that extent that radiation of a certain frequency can only appear in packets of certain amounts of energy. Next confirmation was due to Albert Einstein, who explained the photoelectric effect using the Planck's rule (2.1). Einstein was the first to set forward the concept "quantum of light" or "quantum of energy". Now it became clear that Planck's law was not a property of the oscillators, but a property of the radiation itself (and we are talking about electromagnetic radiation, the early quantum mechanics was only concerned on that). Nevertheless, there is also a quantization of the oscillators themselves, so that they can only absorb, and emit, certain distinct levels of energy, as will be seen below.

Although the energy quantization was at first only a tentative idea, it was soon realized that quantum physics was here to stay. One thing logically lead to an other, and in end most of the physics got to be rewritten into the modern quantum language. Not only is energy quantized, but almost everything in physics. What is not yet shown to be quantized sooner or later might be. I do not intent to explain in detail the historical development of quantum theory, it is readily explained elsewhere, in books and in Internet. I will in stead use a simplified quantum object, the linear harmonic oscillator, in order to demonstrate the ideas and mathematical methods of quantum theory.

## 2.2 The Simple Harmonic Quantum Oscillator

Consider an microscopic one-dimensional harmonic oscillator e.g. a diatomic molecule. We take from the classical case the basic constituents. We need the reduced mass  $m$ , and the angular velocity, which is simply related to the oscillation frequency  $\omega = 2\pi\nu$ , and in stead of velocity we are going to use the linear momentum  $p = m v$ .

$$\text{Potential energy:} \quad V(x) = \frac{1}{2} k x^2 = \frac{1}{2} m \omega^2 x^2 \quad (2.3)$$

$$\text{Kinetic energy:} \quad T = \frac{1}{2} m v^2 = \frac{p^2}{2m} \quad (2.4)$$

Hamiltonian function, from the total energy:

$$\mathcal{H}(p, x) = T + V = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \quad (2.5)$$

## 2.3 The Correspondence Principle

We now go over to a quantum mechanical system using the "correspondence principle", which means that going from macroscopic to microscopic objects, the numerical algebra of classical mechanics is taken over by operator algebra in quantum mechanics. These operators are differential operators. It is possible, in fact, to use an operator algebra also in classical mechanics. This kind of mathematical tools are often used in solving of differential equations, and also the equation (1.8) of the harmonic oscillator is a differential equation. It is only that in quantum mechanics these operators must be chosen in a special way in order to give the expected quantized results.

1. The position  $x$  becomes the position operator, which simply consist of a multiplication with the value  $x$ . This is a very elementary operator, but formally we need to call it an operator.

$$\mathbf{x} = x \quad (2.6)$$

2. The linear momentum  $p$  becomes the differential operator (the mathematical logic of operators needs to use imaginary numbers,  $i$  denotes the imaginary unit:  $i = \sqrt{-1}$ )

$$\mathbf{p} = -i \hbar \frac{\partial}{\partial x} \quad (2.7)$$

It is customary to use the "slashed h" symbol, the Dirac constant, in stead of Plank's constant  $h$ . This shortens the expressions somewhat in many cases

$$\hbar = \frac{h}{2\pi} \quad (2.8)$$

3. The Hamiltonian function  $\mathcal{H}$  becomes now the Hamiltonian operator  $\mathbf{H}$ :

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2} m \omega^2 \mathbf{x}^2 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \quad (2.9)$$

The operators operate on a function, of historical reasons called the "wave equation". Formally mathematically it is a generating function, because operating on it gives the physical quantities the operators represent. The correspondence principle and going over to differential operators is not that far-fetched as it might look at first sight, also the classical theoretical physics could be, and had been, formulated using operators and generating functions. These two operators above obey canonical correspondence principle (position and linear momentum are what is called "canonical conjugates" in the classical theory):

$$\begin{aligned}
 (\mathbf{x} \mathbf{p} - \mathbf{p} \mathbf{x}) \Psi &= \mathbf{x} \left( -i \hbar \frac{\partial \Psi}{\partial \mathbf{x}} \right) - \left[ -i \hbar \frac{\partial}{\partial \mathbf{x}} (\mathbf{x} \Psi) \right] = \dots \\
 \dots &= -i \hbar \mathbf{x} \frac{\partial \Psi}{\partial \mathbf{x}} + i \hbar \mathbf{x} \frac{\partial \Psi}{\partial \mathbf{x}} \Psi + i \hbar \frac{\partial \mathbf{x}}{\partial \mathbf{x}} \Psi = i \hbar \Psi
 \end{aligned}
 \tag{2.10}$$

and so the commutator (customarily denoted with the square parenthesis) gives

$$[\mathbf{x}, \mathbf{p}] = \mathbf{x} \mathbf{p} - \mathbf{p} \mathbf{x} = i \hbar
 \tag{2.11}$$

## 2.4. The Schrödinger Equation

Erwin Schrödinger (German physicist 1887 - 1961) was the first to suggest a method setting up the quantization condition. If we have a system that does not change in time (other than periodic changed, that is oscillations) we have the time-independent form of the Schrödinger equation (otherwise there is a time derivative of  $\Psi$  too in the equation). The method consists of using the Hamiltonian operator and claiming that there should be a constant, the total energy, that gives same result as operating with the Hamiltonian operator. The Hamiltonian operator  $\mathbf{H}$  is the one with the property that it produces the energy values  $E$  in that way, by operating on the wave-function.

$$\mathbf{H} \Psi = E \Psi
 \tag{2.12}$$

The Schrödinger method always gives a differential equation to solve. The Schrödinger equation for the harmonic oscillator is a differential equation in such time-independent form:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial \mathbf{x}^2} + \frac{1}{2} m \omega^2 \mathbf{x}^2 \Psi = E \Psi
 \tag{2.13}$$

where  $E$  is the oscillators total energy. This is a second grade differential equation, and such equations also appear in classical theory of wave movement (like waves in water). Of this formal reason (it is not more mystical) the function  $\Psi$  is called the "wave function". The differential equation (2.13) can be solved using some standard method for solving of differential equations, like the power series method (found in mathematical books, it is interesting mathematically but somewhat tedious, therefore only the result is given here). The resulting quantum mechanical "wave function", is a quantized function, meaning that it depends on a parameter that can only take a discrete set of values (integers 0,1, ... in this case). The wave function for the harmonic oscillator becomes (this is in normalized form)

$$\Psi_n(x) = A e^{-\frac{(\alpha x)^2}{2}} H_n(\alpha x)
 \tag{2.14}$$

where 
$$\alpha = \sqrt{\frac{m \omega}{\hbar}} \tag{2.15}$$

and the normalizing constant is (explained below) 
$$A = \sqrt{\frac{\alpha}{2^n n! \sqrt{\pi}}} \tag{2.16}$$

and the Hermite polynomial which appears in the expression (2.14) is defined as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) \tag{2.17}$$

(Not to be confused with the Hamiltonian, there are many H:s in different forms here.) It is a polynomial function which is part of the function  $\Psi$ , which we in (2.14) denote  $\Psi_n$ , because there is more than one possible solution. In fact there is a solution corresponding every nonnegative integer  $n$ . Because  $n$  can only take discrete values, integers, it is a quantum number. For every possible value for  $n$  there is a corresponding quantized energy (not proved here, but we do it later on):

$$E_n = \left( \frac{1}{2} + n \right) \hbar \omega \quad n = 0, 1, 2, \dots \tag{2.18}$$

## 2.5. Heisenberg's Uncertainty Principle

Werner Heisenberg (German physicist 1901-1976) developed an other method (independently and somewhat earlier than Schrödinger) called the matrix method. Heisenberg set up an equation with the formal appearance like (2.12), but the operator there is a matrix operator  $\mathbf{H}$  that operates a vector  $\Psi$ . A Heisenberg matrix is a formal mathematical object, a quadratic table of numbers, with special multiplication rules, that operating on vectors gives the desired quantized energy values. This kind of matrix equation is called an eigenvalue (German for own-value or specific value) equation. The corresponding vector  $\Psi$  is called the eigenvector or state vector. Matrices were used before Heisenberg in solving of groups of linear equations, but Heisenberg, not knowing of it, in effect reinvented the matrices for quantum mechanical purposes.

It was soon found out that there were a logical connection between Schrödinger's and Heisenberg's methods. It was possible to express the differential operators as matrices in respect to the energy values in similar manners, and the results from the both theories were identical. These two methods therefore are now seen as two sides of one and the same mathematical reality. They are still, of historical reasons, often called the "wave mechanics" (Schrödinger's) and the "matrix mechanics" (Heisenberg's). This is also called the "complementarily principle", or the "wave-matrix- complementarily".

The commutation rule (2.11) also holds for matrix operators. Starting from it Heisenberg deduced the uncertainty relations

$$\Delta p \Delta x \leq \frac{1}{2} \hbar \tag{2.19}$$

$$\Delta E \Delta t \leq \frac{1}{2} \hbar \tag{2.20}$$

This leads to the probabilistic nature of predicted measurements in quantum physics. A measurement is given as an expectation value. In the commonplace mathematical statistics an expectation value is defined as the integral:

$$\langle f(x) \rangle = \int_{x_0}^{x_1} f(x) P(x) dx \quad (2.21)$$

where  $P(x)$  is the probability distribution function for the statistical variable  $x$ . This is for real valued functions  $P$ . In quantum mechanics we have instead the complex valued wave function  $\Psi$ , and a physical "observable" represented by an operator  $f(x)$ . The expectation value, which still must be realvalued:

$$\langle f(x) \rangle = \int_{x_0}^{x_1} \Psi^* f \Psi dx \quad (2.22)$$

Here  $\Psi^*$  is the complex conjugate of  $\Psi$ . The expectation values for position, momentum, and energy become respectively:

$$\langle x \rangle = \int_{x_0}^{x_1} \Psi^* x \Psi dx = \int_{x_0}^{x_1} \Psi^* \cdot x \Psi dx = \int_{x_0}^{x_1} x |\Psi|^2 dx \quad (2.23)$$

$$\langle p \rangle = \int_{x_0}^{x_1} \Psi^* p \Psi dx = \int_{x_0}^{x_1} \Psi^* \cdot \frac{\hbar}{i} \frac{\partial}{\partial x} \Psi dx \quad (2.24)$$

$$\langle E \rangle = \frac{(\langle p \rangle)^2}{2m} = \int_{x_0}^{x_1} \Psi^* p^2 \Psi dx = \int_{x_0}^{x_1} \Psi^* \cdot \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi dx \quad (2.25)$$

Integrated over the values of separation  $x$  from the smallest possible value  $x_0$  to the greatest value  $x_1$ , which (formally mathematically) might limit to infinity. The definition (2.21) from mathematical statistics is a special case of the quantum mechanical (2.22), where the operation is simply a multiplication with a real number, like it is in (2.23). We see then that the probability distribution corresponds to the square of the wave function

$$P(x) = \Psi^* \Psi = |\Psi|^2 \quad (2.26)$$

Such a distribution must be normalized to one, saying that the distribution covers all possible cases, but no more than that. Also the normalizing constant (2.16) above was based to such condition, that is, if the condition below is not met the function is multiplied with a suitable constant in order to make it meet the condition:

$$\int_{x_0}^{x_1} \Psi^* \Psi dx = 1$$

This makes clear the point that the wave function  $\Psi$  (more precisely, its square) is an expression of the probability distribution for the measured physical value.

In the Heisenberg's formulation of quantum mechanics  $\Psi$  is a (formal mathematical) vector, the quantum state vector, and probability density is the square length of this (formal) vector. Note also that the mathematical structure of quantum mechanics is that of a linear algebra, a general mathematical theory that can be used as well for matrix and vector calculus as for solving of many kind differential equations.

## 2.7. Dirac's Bra-Ket Formulation

Dirac (English physicist 1902 - 1984) set forward a mathematical method that combined Schrödinger's and Heisenberg's methods into one logical structure. For that purpose he invented new mathematical symbols, now known as Dirac's bra-ket formulation. It is today the standard formulation of quantum physics. Starting again from the Schrödinger equation, the quantified solutions and energies written in the equation:

$$\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_n}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \Psi_n = E_n \Psi \quad (2.27)$$

The Schrödinger equation can be represented by operator eigenstate equation. Because we are dealing with energies, we have here the Hamiltonian operator  $\mathbf{H}$ , which has energies as eigenvalues. Every one of the wave function solutions corresponds to a state vector in a Hilbert space of functions (a kind of linear algebra). Using the Dirac "bra-ket" notation, written as an eigenvector "ket"  $|\Psi_n\rangle$

$$\mathbf{H} |\Psi_n\rangle = E_n |\Psi_n\rangle \quad (2.28)$$

The wave-function now expresses the probability density function for finding an oscillator with energy state  $n$  at position value  $x$ :

$$\Psi_n(x) = \langle x | \Psi_n \rangle \quad (2.29)$$

Note: vector algebraically it is an inner product of a "bra" vector and a "ket" vector. "Bra" corresponds to a row-vector and "ket" to a column-vector. Generally expressions in form  $\langle x | y \rangle$ , like that in (2.29) can be treated as inner products in a formal mathematical function space, called Hilbert's space. The formulation of quantum theory follows the outlines of such mathematical theory.

The energy eigenvalues for the Hamiltonian operator are

$$E_n = E_0 + n \hbar \omega \quad n = 0, 1, 2, \dots \quad (2.30)$$

where the lowest energy  $E_0 = \frac{\hbar \omega}{2}$  (2.31)

is the "ground energy" or "zero-point energy": a quantum oscillator always oscillates with minimum of this energy, it can never "freeze" and stand still in zero energy.

Note: it is customary to write the  $n$ :th state ket  $|\Psi_n\rangle$  simply as  $|n\rangle$ . The state ket corresponding to the ground state  $|\Psi_0\rangle$  is then written  $|0\rangle$ . Ground state is not the same as "zero ket" or "empty state", written  $|\rangle$ , with no physical objects (oscillators) present.

## 2.8. The Ladder Operator Method

A somewhat more abstract but very effective way in solving the problem of eigenstates and eigenvalues is using ladder operators. This formulation has also the advantage that it can be further generalized to Quantum Electrodynamics, and other quantum field theories, that are currently the most advanced generally accepted theories.

Define an operator expressed in form:

$$\mathbf{a} = A (\mathbf{x} + i B \mathbf{p}) \quad (2.32)$$

where the two constants are defined:

$$A = \sqrt{\frac{m \omega}{2 \hbar}} = \frac{\alpha}{\sqrt{2}} \quad B = \frac{1}{m \omega} = \frac{1}{\hbar^2 \alpha^2} \quad (2.33)$$

As for a complex number  $z = a + i b$  there is the complex conjugate  $z^* = a - i b$ , quite similarly for an operator there is a corresponding "Hermitian adjoint" or "Hermitian conjugate". It is defined as the corresponding operator that has its eigenvalues complex conjugate to the original one. It can also be effectively defined using the Dirac bra-ket:

$$\langle A x | y \rangle = \langle x | A^\dagger y \rangle \quad (2.33)$$

For an operator given in form (2.32) the Hermitian conjugate can be written

$$\mathbf{a}^\dagger = A (\mathbf{x} - i B \mathbf{p}) \quad (2.34)$$

Summing of the operator and its adjoint gives

$$\mathbf{a}^\dagger + \mathbf{a} = A (\mathbf{x} + i B \mathbf{p}) + A (\mathbf{x} - i B \mathbf{p}) = 2 A \mathbf{x} \quad (2.35)$$

so we see that the displacement operator is now

$$\mathbf{x} = \frac{1}{2 A} (\mathbf{a}^\dagger + \mathbf{a}) = \sqrt{\frac{\hbar}{2 m \omega}} (\mathbf{a}^\dagger + \mathbf{a}) \quad (2.36)$$

and subtracting them

$$\mathbf{a}^\dagger - \mathbf{a} = A (\mathbf{x} + i B \mathbf{p}) - A (\mathbf{x} - i B \mathbf{p}) = -i 2 A B \mathbf{p} \quad (2.37)$$

so we see that the momentum operator is

$$\mathbf{p} = \frac{i}{2 A B} (\mathbf{a}^\dagger - \mathbf{a}) = \sqrt{\frac{\hbar m \omega}{2}} (\mathbf{a}^\dagger - \mathbf{a}) \quad (2.38)$$

Remembering that the operators do not commute, we get

$$\begin{aligned} \mathbf{a}^\dagger \mathbf{a} &= A (\mathbf{x} - i B \mathbf{p}) A (\mathbf{x} + i B \mathbf{p}) = A^2 (\mathbf{x}^2 + i B \mathbf{x} \mathbf{p} - i B \mathbf{p} \mathbf{x} + B^2 \mathbf{p}^2) = \dots \\ &\dots = A^2 (\mathbf{x}^2 + B^2 \mathbf{p}^2 + i B \cdot [\mathbf{x}, \mathbf{p}]) = A^2 \mathbf{x}^2 + A^2 B^2 \mathbf{p}^2 - A^2 B \hbar \end{aligned} \quad (2.39)$$



and

$$\begin{aligned} \mathbf{a} \mathbf{a}^\dagger &= A (\mathbf{x} + i B \mathbf{p}) A (\mathbf{x} - i B \mathbf{p}) = A^2 (\mathbf{x}^2 - i B \mathbf{x} \mathbf{p} + i B \mathbf{p} \mathbf{x} + B^2 \mathbf{p}^2) = \dots \\ &\dots = A^2 (\mathbf{x}^2 + B^2 \mathbf{p}^2 - i B \cdot [\mathbf{x}, \mathbf{p}]) = A^2 \mathbf{x}^2 + A^2 B^2 \mathbf{p}^2 + A^2 B \hbar \end{aligned} \quad (2.40)$$

It is easily seen from (2.39) and (2.40) that we have for the constant expressions above

$$A^2 = \frac{m \omega}{2 \hbar} \quad A^2 B^2 = \frac{1}{2 \hbar m \omega} \quad A^2 B = \frac{1}{2 \hbar} \quad (2.41)$$

and substituting these we get

$$\mathbf{a}^\dagger \mathbf{a} = \frac{m \omega}{2 \hbar} \mathbf{x}^2 + \frac{1}{2 \hbar m \omega} \mathbf{p}^2 - \frac{1}{2} \quad (2.42)$$

$$\mathbf{a} \mathbf{a}^\dagger = \frac{m \omega}{2 \hbar} \mathbf{x}^2 + \frac{1}{2 \hbar m \omega} \mathbf{p}^2 + \frac{1}{2} \quad (2.43)$$

so the commutator becomes

$$[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{a} \mathbf{a}^\dagger - \mathbf{a}^\dagger \mathbf{a} = 1 \quad (2.44)$$

Now, from the first of the products we get

$$\mathbf{a}^\dagger \mathbf{a} = \frac{1}{\hbar \omega} \left( \frac{1}{2} m \omega^2 \mathbf{x}^2 + \frac{\mathbf{p}^2}{2 m} \right) - \frac{1}{2} = \frac{1}{\hbar \omega} \mathbf{H} - \frac{1}{2} \quad (2.45)$$

So we can write the Hamiltonian in form

$$\mathbf{H} = \hbar \omega \left( \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \right) \quad (2.46)$$

If the energy eigenstate (to the Hamiltonian operator) with energy E is written as the Dirac "ket"  $|\Psi_E\rangle$ , then operating it with the ladder operator  $\mathbf{a}$  is written  $\mathbf{a} |\Psi_E\rangle$ , and the adjoint of this is expressed as operating the corresponding Dirac "bra" from right with the adjoint operator:  $\langle \Psi_E | \mathbf{a}^\dagger$ . An inner product of vectors v and u, operated by X and Y, is generally written in the "bra-ket" formulation:

$$(X v, Y u) = \langle v | X^\dagger Y | u \rangle \quad (2.47)$$

For the inner product of the operation above with itself we have

$$(\mathbf{a} |\Psi_E\rangle, \mathbf{a} |\Psi_E\rangle) = \langle \Psi_E | \mathbf{a}^\dagger \mathbf{a} | \Psi_E \rangle \geq 0 \quad (2.48)$$

Using the result shown earlier we can write

$$\langle \Psi_E | \frac{1}{\hbar \omega} \mathbf{H} - \frac{1}{2} | \Psi_E \rangle = \left( \frac{1}{\hbar \omega} E - \frac{1}{2} \right) \langle \Psi_E | \Psi_E \rangle = \frac{E}{\hbar \omega} - \frac{1}{2} \geq 0 \quad (2.49)$$

so that 
$$E \geq \frac{\hbar \omega}{2} \quad (2.50)$$

When the transformed ket  $\mathbf{a} |\Psi_E\rangle$  is a zero-ket (ground state), then we have the lowest energy

$$E_0 = \frac{\hbar \omega}{2} \quad (2.51)$$

Next calculate the commutators with Hamiltonian

$$\begin{aligned} [\mathbf{H}, \mathbf{a}] &= \mathbf{H} \mathbf{a} - \mathbf{a} \mathbf{H} = \left[ \left( \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \right) \mathbf{a} - \mathbf{a} \left( \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \right) \right] \hbar \omega = \dots \\ \dots &= \left( \mathbf{a}^\dagger \mathbf{a} \mathbf{a} + \frac{1}{2} \mathbf{a} - \mathbf{a} \mathbf{a}^\dagger \mathbf{a} - \frac{1}{2} \mathbf{a} \right) \hbar \omega = -\hbar \omega [\mathbf{a}, \mathbf{a}^\dagger] \mathbf{a} = -\hbar \omega \mathbf{a} \end{aligned} \quad (2.52)$$

And similarly can be found out

$$[\mathbf{H}, \mathbf{a}^\dagger] = \hbar \omega \mathbf{a}^\dagger \quad (2.53)$$

When  $\mathbf{a} |\Psi_E\rangle$  is not a zero ket (empty ket)

$$\begin{aligned} \mathbf{H}(\mathbf{a} |\Psi_E\rangle) &= (\mathbf{H} \mathbf{a}) |\Psi_E\rangle = (\mathbf{H} \mathbf{a} - \mathbf{a} \mathbf{H} + \mathbf{a} \mathbf{H}) |\Psi_E\rangle = \dots \\ \dots &= ([\mathbf{H}, \mathbf{a}] + \mathbf{a} \mathbf{H}) |\Psi_E\rangle = (-\hbar \omega \mathbf{a} + \mathbf{a} \mathbf{E}) |\Psi_E\rangle = (\mathbf{E} - \hbar \omega) (\mathbf{a} |\Psi_E\rangle) \end{aligned} \quad (2.54)$$

Similarly

$$\mathbf{H}(\mathbf{a}^\dagger |\Psi_E\rangle) = (\mathbf{E} + \hbar \omega) (\mathbf{a}^\dagger |\Psi_E\rangle) \quad (2.55)$$

So, taking an energy eigenstate with energy eigenvalue E. operator  $\mathbf{a}$  acting on it lowers it to the nearest lower energy state. Correspondingly the adjoint operator  $\mathbf{a}^\dagger$  rises it to the nearest higher energy state. We say that

$\mathbf{a}$  is lowering or *annihilation operator*, also written as  $\mathbf{a}_-$ .

$\mathbf{a}^\dagger$  is rising or *creation operator*, also written as  $\mathbf{a}_+$ .

Note: the lowering operator operating on the ground state ket leaves "an empty state", and so gives the zero ket (meaning not zero energy but "no physical system present"). It is also called the "vacuum state", because in quantum field theories it represents the vacuum.

$$\mathbf{a} |0\rangle = | \rangle$$

The ground state ket  $|0\rangle$  itself is not an empty state, but corresponds to the lowest energy state, where we have

$$\mathbf{H} |0\rangle = E_0 |0\rangle$$

The zero-point (or ground state) energy

$$E_0 = \frac{\hbar \omega}{2} > 0$$

is the energy-eigenvalue corresponding the ground state.