The theory of neutrino oscillations

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Introduction

The intention in this text is to give an elementary quantum mechanical presentation and explanation to a quite recent observation, namely that neutrinos appear to change their identity during time. The full theory has three neutrino states, but I will first demonstrate the principle in a simpler two-neutrino-state picture, which also turns out to have practical applications in approximate cases. This two-neutrino model is sometimes given as an example on lower level courses of quantum mechanics. It is a nice one because it is "bleeding edge" particle physics, but still only very basic quantum mechanical techniques are involved.

The presentation keeps on quite an elementary quantum-mathematical level. Though, the reader is expected to have some basic knowledge of the mathematics of quantum mechanics, and matrix and linear algebras. The only thing that might seem to be more advanced is maybe the Dirac notation, but it is merely an other way of writing the commonplace linear and matrix algebras.

What are neutrinos?

Neutrinos as particles that belong to the lepton-family. The other leptons are: electron, muon, and taon. The same way as these other leptons are three, there are known to exist three kind of neutrinos, three "flavors", called electron neutrino, muon neutrino, and taon neutrino. They are very feeble and react extremely seldom with other particles. Their existence was first established indirectly from particle decay experiments. Recently large neutrino detectors have constructed that can catch a small fraction of the total of a large neutrino flow, so that there now are direct measurements of neutrino reactions available.

Neutrinos were for a long time considered as massless particles. The phenomenon of neutrino oscillation is now generally accepted and experimentally verified. It implies, which first may seem an unexpected connection, that neutrinos have a mass, although very small. (Note: an older nomenclature used the term "rest-mass" for what currently is called simply "mass".) The oscillation theory suggests that neutrinos do not keep their flavor, but slowly change from one type to an other. Among other things this would explain why electron neutrinos from Sun that hit Earth are far lesser in number than what is expected from the knowledge of the nuclear reactions in Sun.

Historical background to the presented theory

Although the results that prove neutrino oscillation are quite recent, the theory has existed quite a while. It was first described by Bruno Pontecorvo and Vladimir Gribov 1969, based on a work that Pontecorvo had started at 1950:ies. The theory was more recently improved by Ziro Maki, Masami Nakagawa, and Shoichi Sakata.

The simplified two-neutrino model

Consider 2 neutrinos ν_α and ν_β (take any two of ν_e, ν_μ, and ν_τ). Choosing a simple vector form for the calculations, the lepton-number eigenstates (the original neutrinos) expressed in 2-vector representations (an expression like "$|\,..\rangle$" denotes a quantum state vector in Dirac's "bra-ket" notation):

$$|\,\nu_\alpha\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad\qquad |\,\nu_\beta\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{1}$$

These now represent "pure" neutrino sates, the corresponding lepton numbers, also called lepton "flavors", are α =1 and β = 0 for the first one, α = 0 and β = 1 for the second one. If for example α denotes μ and β denotes τ, the first one of the state vectors (1) represent a μ-neutrino, and the second a τ-neutrino. Generally in quantum mechanics, before any measurement is performed, we can have a mixed state. Write a mixed quantum state of these two neutrino states as follows

$$|\,\nu\,\rangle = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \tag{2}$$

Here a and b denote some mixing coefficients, generally complex numbers. The probability for a certain neutrino to be detected in a measurement:

$$\begin{aligned} P_\alpha &= \left|\langle\nu_\alpha|\nu\rangle\right|^2 = \left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}\right|^2 = |a|^2 \\ P_\beta &= \left|\langle\nu_\beta|\nu\rangle\right|^2 = \left|\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}\right|^2 = |b|^2 \end{aligned} \tag{3}$$

Note: probability is given by modulus square of a complex-valued probability amplitude, in Dirac notation $\langle\,..|\,..\,\rangle$.

We assume in this two-neutrino (approximate) case that (1) is a complete representation (a complete base in the corresponding Hilbert space of quantum states). Thus the two probabilities of the both alternatives got to sum up to 1, which also is granted because we always expect the state-vectors (2) be normalized:

$$P_\alpha + P_\beta = |a|^2 + |b|^2 = 1 \tag{4}$$

This is the usual quantum logic of mixed states: it is not possible, even in principle, in beforehand establish which of the neutrinos that will hit the detector is of which kind, but quantum theory still is expected to give the right prediction for the detected proportions of each kind of neutrino.

In the standard theory of particle physics the neutrino states were <u>not</u> expected to change in time, and thus the detected proportions should remain the same independent of how long away from the source the detection was performed. However, experiments showing a growing evidence against that point of view, the conclusion became that there must be some kind of time dependency. The following theory gives a simple model for it, and is called "neutrino oscillation".

The basic assumptions in the neutrino oscillation theory

The basic assumptions are:

1. *neutrinos have mass*
2. *the mass eigenstates are not identical to the lepton number eigenstates*

The existence of a mass (rest-mass) is, as we will see, a necessary condition in order to have mass eigenstates that are not identical with the flavor eigenstates. The mass eigenstates are then mixed states in the above flavor state representation (1), here denoted by

$$| \nu_m \rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5}$$

This is, in matter of fact, quite natural. Because the measurement of the neutrino flavor is a different operation from the measurement of the neutrino mass, they do not need to share common eigenvectors. This also tells us that we can not measure the mass of any of the neutrinos, electron, muon or taon neutrino, as we express them with the flavors. Only such mixed neutrino flavor states that are mass eigenstates can be involved in a mass measurement.

We could, of course, try to measure for example the mass of an electron neutrino, but the result would be the mass of some of the two mass eigenstates. When trying to find out which of the neutrinos corresponds to the masses gives flavors randomly with certain probability distributions, actually those given by (3). Or if the measurement is not accurate enough to separate between the two masses (the difference is very slight), it will give an average mass, which is what usually is meant when neutrino masses are given in particle lists.

Transformation between flavor and mass states

We have here two base vectors, so the transformation between the lepton number representation and the mass-representation can now be represented by a 2x2 matrix:

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{6}$$

The inverse matrix:

$$\mathbf{M}^{-1} = \frac{1}{M} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \tag{7}$$

where we have the determinant

$$M = |\mathbf{M}| = M_{11}\, M_{22} - M_{12}\, M_{21} \tag{8}$$

The determinant is usually set to 1 (the matrix is assumed unimodular), but we do not need it here.

Definition of Hermitian adjoint of a matrix operator is the complex conjugate of the transpose matrix (* denotes complex conjugation) :

$$\mathbf{M}^{\dagger} = \left(\mathbf{M}^{T}\right)^{*} = \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}^{*} = \begin{pmatrix} M_{11}^{*} & M_{21}^{*} \\ M_{12}^{*} & M_{22}^{*} \end{pmatrix} \tag{9}$$

Generally the transformation between the mass states (index m) and flavor states (index f) is

$$|\nu_m\rangle = \mathbf{M}|\nu_f\rangle \tag{10}$$

$$|\nu_f\rangle = \mathbf{M}^{\dagger}|\nu_m\rangle \tag{11}$$

In matrix form (capital letters for the mass state vectors)

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} M_{11}^{*} & M_{21}^{*} \\ M_{12}^{*} & M_{22}^{*} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{13}$$

Assume a neutrino starting as one with flavor α , that means that the start state is:

$$|\,\nu_f\,,0\,\rangle = |\,\nu_\alpha\,\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14}$$

In terms of the mass eigenstates the start state is expressed

$$|\,\nu_m\,,0\,\rangle = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix} \tag{15}$$

The start state then, even if it a pure one flavor state and contains only one kind of neutrinos (let's say electron neutrinos), is still a mix of the two mass states.

Hamiltonian and the time evolution operator

We assume free flying neutrinos (there are also corresponding theories of neutrinos inside stars etc.) , and in case of a free particle we have the time dependency of a mass state

$$|\,\nu_m\,,t\,\rangle = \mathbf{U_m}(t)\;|\,\nu_m\,,0\,\rangle \tag{16}$$

where the free-particle time evolution operator is generally of form

$$\mathbf{U_m}(t) = e^{-i\;\mathbf{H}\;t} \tag{17}$$

Here $\mathbf{H}$ is the Hamiltonian of the neutrino system. It has eigenvalues E_1, E_2, which are the total energies of the system, including both mass-energy and the kinetic energy contribution.

If we now (in contrary to what is done in (1)) choose the representation of the two mass eigenstates as follows

$$|\,\nu_1, t\,\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_m \qquad |\,\nu_2, t\,\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_m \tag{18}$$

which we are free to do, because the matrix representation is only a mathematical tool, but we mark the mass state vectors with index m to not to confuse them with the flavor state eigenvectors. These are then eigenstates of the time independent free-particle Hamiltonian, which implies that the Hamiltonian matrix in the mass state representation (18) is of diagonal form

$$\mathbf{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \tag{19}$$

Because the neutrinos are expected to move with relativistic speeds we get the energies from:

$$E_i = \sqrt{m_i^2\, c^4 + p_i^2\, c^2} \tag{20}$$

where i = 1 or 2 , and p_i are the corresponding linear moments. When the masses differ, then generally also the energies. For practical calculations neutrinos can be treated as nearly speed-of-light particles. The energies can then be approximated as:

$$E_i \approx \left(E + \frac{m_i\, c^2}{2\, E} \right) \tag{21}$$

where E is the energy for a velocity-of-light particle (as neutrinos would be without the mass). The second term here differs and makes Hamiltonian (19) non degenerate (different E_i :s), which is necessary for the flavor oscillations, as we soon will see. We do not need to use the energy formulas (20) or (21) explicitly in the following, but use directly the energy eigenvalues E_i, whatever way they might be given. It turns out that only the differences between E_1 and E_2 are what matter in the end result.

The time evolution operator in matrix form becomes in the mass representation:

$$\mathbf{U_m}(t) = \begin{pmatrix} e^{-i\, E_1\, t} & 0 \\ 0 & e^{-i\, E_2\, t} \end{pmatrix} \tag{22}$$

But in (1) we used the flavor representation. In order to use the time evolution operator on them, we must first transform it to the flavor base representation. This is done by the similarity transformation:

$$\mathbf{U_f}(t) = \mathbf{M}^{-1}\, \mathbf{U_m}(t)\, \mathbf{M} \tag{23}$$

We could start transforming the neutrino flavor states to mass states and then applying the time evolution, and then transforming back to the flavor states. It is the deduction often seen. A mathematically equivalent way, but maybe "pedagogically" more satisfactory, is by first transforming the time evolution operator, and operating with it directly the flavor states. After all, it is the time evolution of a neutrino of a given flavor we want to investigate.

Time evolution of the mass states

The neutrino oscillation, as will be shown, is purely due to the time evolution of the flavor states. In contrary, when time evolution is applied to the mass states, the measured masses do not change. It is easy to see, the time evolution of a mass state (we take the first mass eigenstate) becomes:

$$|\,\nu_1, t\,\rangle = \mathbf{U_m}(t)\ |\,\nu_1, 0\,\rangle = \begin{pmatrix} e^{-i E_1 t} & 0 \\ 0 & e^{-i E_2 t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_m e^{-i E_1 t} \tag{24}$$

The mass state expectation value, the probability for having the same mass after time t, becomes the square of the following

$$\langle\,\nu_1, t\,|\,\nu_1, t\,\rangle = (1 \quad 0)_m\ e^{i E_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_m e^{-i E_1 t} = 1 \tag{25}$$

Similarly we could show it to the other mass eigenstate. It is then clear that the masses do not oscillate.

Note: It is crucial that the reasoning is done fully in the quantum way. Using a semiclassical reasoning one could easily come to the incorrect conclusion that the neutrinos, when they change their appearance, also change their masses, which would break against the conservation of the mass-energy. The quantum mechanical reasoning rather gives an opposite point of view: in order to keep the conservation of mass-energy the neutrino flavors are bound to oscillate.

Time evolution of the flavor state

Performing the matrix operations in (23), the result becomes

$$\mathbf{U_f}(t) = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \tag{26}$$

where the components are

$$U_{11} = \frac{M_{11}\,M_{22}}{M}\,e^{-iE_1} - \frac{M_{21}\,M_{12}}{M}\,e^{-iE_2\,t}$$

$$U_{22} = -\frac{M_{12}\,M_{21}}{M}\,e^{-iE_1\,t} + \frac{M_{22}\,M_{11}}{M}\,e^{-iE_2\,t}$$

$$U_{12} = \frac{M_{12}\,M_{22}}{M}\,e^{-iE_1\,t} - \frac{M_{22}\,M_{12}}{M}\,e^{-iE_2\,t} \tag{27}$$

$$U_{21} = -\frac{M_{11}\,M_{21}}{M}\,e^{-iE_1\,t} + \frac{M_{21}\,M_{11}}{M}\,e^{-iE_2\,t}$$

Thus for a given pure one-flavor-only neutrino state (that we now here denote with α) at time t :

$$|\,\nu_\alpha, t\,\rangle = \mathbf{U_f}(t)\;|\,\nu_\alpha, 0\,\rangle = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}U_{11}\\U_{21}\end{pmatrix} = U_{11}\begin{pmatrix}1\\0\end{pmatrix} + U_{21}\begin{pmatrix}0\\1\end{pmatrix} \tag{28}$$

$$|\,\nu_\beta, t\,\rangle = \mathbf{U_f}(t)\;|\,\nu_\beta, 0\,\rangle = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}U_{12}\\U_{22}\end{pmatrix} = U_{12}\begin{pmatrix}1\\0\end{pmatrix} + U_{22}\begin{pmatrix}0\\1\end{pmatrix} \tag{29}$$

Here $U_{ij} = U_{ij}(t)$ are time dependent. The former state seemingly contains the β-neutrino state at time t > 0, the later the α-neutrino state, although the states were assumed as one flavor states originally (t = 0). The probability, square modulus of this

$$P_{ij} = \left(\left|U_{ij}\right|\right)^2 = U_{ij}^{\;*}\;U_{ij} \tag{30}$$

1) In the former start condition (α-neutrinos only):
The amplitude of encountering a α-neutrinos at time t > 0

$$\langle\lambda_\alpha|\lambda_\alpha, t\rangle = U_{11}(t)\langle\lambda_\alpha|\lambda_\alpha\rangle + U_{21}(t)\langle\lambda\alpha|\lambda_\beta\rangle = U_{11}(t) \tag{31}$$

The probability becomes

$$P_{11} = U_{11}^{\;*}\;U_{11} = A - \frac{B}{2}\left[e^{i\,(E_2 - E_1)\,t} + e^{-i\,(E_2 - E_1)\,t}\right] = A - B\,\cos\left[(E_2 - E_1)\,t\right] \tag{32}$$

Denoting with the constants A_{11} and B_{11} the constant expressions:

$$A = \frac{M_{11}^{\;2}\,M_{22}^{\;2} + M_{12}^{\;2}\,M_{21}^{\;2}}{M^2} \qquad B = 2\,\frac{M_{11}\,M_{12}\,M_{21}\,M_{22}}{M^2} \tag{33}$$

The amplitude of encountering a β-neutrinos at time $t > 0$

$$\langle \lambda_\beta | \lambda_\alpha, t \rangle = U_{11}(t) \langle \lambda_\beta | \lambda_\alpha \rangle + U_{21}(t) \langle \lambda_\beta | \lambda_\beta \rangle = U_{21}(t) \tag{34}$$

$$P_{21} = U_{21}^{*} U_{21} = C_{21} \left[1 - \frac{1}{2} \left[e^{i (E_2 - E_1) t} + e^{-i (E_2 - E_1) t} \right] \right] = C \left[1 - \cos\left[(E_2 - E_1) t \right] \right] \tag{35}$$

$$C = \frac{2 M_{11}^2 M_{21}^2}{M^2} \tag{36}$$

2) In the later start condition (β-neutrinos only):
The amplitude of encountering a β-neutrinos at time $t > 0$

$$\langle \lambda_\beta | \lambda_\beta, t \rangle = U_{12}(t) \langle \lambda_\beta | \lambda_\alpha \rangle + U_{22}(t) \langle \lambda_\beta | \lambda_\beta \rangle = U_{22}(t) \tag{37}$$

$$P_{22} = U_{22}^{*} U_{22} = A - B \cos\left[(E_2 - E_1) t \right] \tag{38}$$

The amplitude of encountering a α-neutrinos at time $t > 0$

$$\langle \lambda_\alpha | \lambda_\beta, t \rangle = U_{12}(t) \langle \lambda_\alpha | \lambda_\alpha \rangle + U_{22}(t) \langle \lambda_\alpha | \lambda_\beta \rangle = U_{12}(t) \tag{39}$$

$$P_{12} = U_{12}^{*} U_{12} = D \left[1 - \cos\left[(E_2 - E_1) t \right] \right] \tag{40}$$

$$D = \frac{2 M_{12}^2 M_{22}^2}{M^2} \tag{41}$$

Conclusion: the probabilities oscillate.

This is due to the cosine term. A, B, C, and D are constants. The theory gives real oscillations in time that depend on the mass (energy) difference of the two mass-states. The time evolution thus mixes the neutrinos, even if we start from a pure state (as we did above). But it only happens if the mass-eigenstates for the Hamiltonian are different (non-degenerate), and it is true only when the neutrinos have non-zero and different masses.

Angle representation of the transformation matrix

The commonplace expression of the transformation is as a formal rotation matrix in the abstract 2-dimensional space of the neutrino flavors. How one can come to that form is shortly given here:

Assume first that the matrix is unimodular, which means that the determinant should be unity. This is not a restriction in the physical theory, and is easily achieved for any non-degenerate matrix, bsy dividing all the components by the determinant). We thus assume that the determinant (8) is unity

$$M = 1 \tag{42}$$

Good physics (not changing measuring scales) also presumes that a transformation matrix is unitary. Generally for a unitary matrix is true:

$$\mathbf{M}^{-1} = \mathbf{M}^{\dagger} \tag{43}$$

which from (7) , and (9) gives following conditions for the components:

$$M_{22} = M_{11}^{*} \qquad -M_{12} = M_{21}^{*} \qquad -M_{21} = M_{12}^{*} \qquad M_{11} = M_{22}^{*} \tag{44}$$

A general form that satisfies the conditions above:

$$\mathbf{M} = \begin{pmatrix} x & y \\ -y^{*} & x^{*} \end{pmatrix} \tag{45}$$

Where x and y are some complex numbers satisfying the unimodularity condition.

$$x\,x^{*} + y\,y^{*} = |x|^{2} + |y|^{2} = 1 \tag{46}$$

Generally one can express complex numbers as:

$$x = c\,e^{i\,\zeta} \qquad y = s\,e^{i\,\delta} \tag{47}$$

Here c and s are real numbers, ζ and δ are arbitrary complex phases. Unimodularity yields

$$x\,x^{*} + y\,y^{*} = c\,e^{-i\,\zeta}\,c\,e^{i\,\zeta} + s\,e^{-i\,\delta}\,s\,e^{i\,\delta} = c^{2} + s^{2} = 1 \tag{48}$$

The result implies that we can formally write

$$c = \cos\theta \qquad s = \sin\theta \tag{49}$$

where θ is an angle. We can then further write:

$$|x|^{2} + |y|^{2} = \cos^{2}\theta + \sin^{2}\theta = 1 \tag{50}$$

The angle θ is practical, but the actual value is not given by the theory. It must be calculated from measurements. The complex phases ζ and δ can be ignored, they do not affect the neutrino oscillation theory. For a greater generality it is satisfactory to keep one of them, which is done in the full theory (shortly presented a bit longer down here).

The matrix now becomes

$$\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{51}$$

Note: sometimes the matrix is given with the sine functions on the diagonal and cosines on the off diagonal, switching the definitions (49) above. In end the difference becomes only a phase factor: a cosine in the result will become a sine. The motivation seems to be that the matrix in that way can be made Hermitian. It is not necessary. It is a transformation matrix, not an operator of a measurement, so it is fully satisfactory that it is unitary.

This quite usual way of giving the matrix has the drawback that it focuses off from the crucial point, that the oscillation is due to the neutrino masses. The formal constant angle θ is nothing but a commonplace way of expressing a constant parameter needed in the theory. It is not involved in the oscillations, it only gives constant values appearing in the expressions of the probabilities:

$$\begin{aligned} A &= \sin^4\theta + \cos^4\theta \\ B &= -2\sin^2\theta\cos^2\theta \\ C = D &= 2\sin^2\theta\cos^2\theta \end{aligned} \tag{52}$$

Lepton quantum numbers logic for the neutrino mass states

There is a question that sounds confusing for a non-physicist: what comes of the lepton quantum number? As we now know, the neutrino, say an electron neutrino, is a mix of the mass states. What is the quantum logic of lepton numbers according to the mass states? For clarity let α be an electron neutrino, and β be a muon neutrino. Now (in the simplified two-neutrino model) electron an muon neutrinos are mixes of the two mass states:

$$\begin{aligned} |\nu_e\rangle &= \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle \\ |\nu_\mu\rangle &= -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle \end{aligned} \tag{53}$$

The left sides above represent the original neutrinos and has thus corresponding lepton number =1. A naive semiclassical idea would be that the mass states then share the electron and muon lepton numbers in proportion to the linear coefficients. The quantum logic goes a bit differently, as usual we must rely on probabilities.

Assume now a pure mass state. It can be expressed as a mix of flavor states. The two mass states expressed with them become

$$\begin{aligned} |\nu_1\rangle &= \cos\theta\,|\nu_e\rangle + \sin\theta\,|\nu_\mu\rangle \\ |\nu_2\rangle &= -\sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle \end{aligned} \tag{54}$$

The probability amplitude in detecting an electron neutrino in a particle interaction, and thus realizing an electron lepton number = 1 , would become:

$$\langle \nu_e | \nu_1 \rangle = \cos\theta \underbrace{\langle \nu_e | \nu_e \rangle}_{=1} + \sin\theta \underbrace{\langle \nu_e | \nu_\mu \rangle}_{=0} = \cos\theta \tag{55}$$

because the states are orthonormal. Similarly for detecting muon neutrino number = 1 :

$$\langle \nu_\mu | \nu_1 \rangle = \cos\theta \underbrace{\langle \nu_\mu | \nu_e \rangle}_{=0} + \sin\theta \underbrace{\langle \nu_\mu | \nu_\mu \rangle}_{=1} = \sin\theta \tag{56}$$

etc.

The conclusion is, that the proportional sharing of the lepton numbers holds, as long as we are talking about the probability amplitudes of detecting them, and not about the lepton quantum numbers themselves (which, after all, must always be integers). The probability is the modulus square of the corresponding amplitude:

$$P(\text{mass\_state\_1}, \text{electron\_lepton\_nr} = 1) = \cos^2\theta \tag{57}$$

$$P(\text{mass\_state\_1}, \text{muon\_lepton\_nr} = 1) = \sin^2\theta$$

etc.

Experimental results for the two-neutrino case

Atmospheric (source mainly from μ-decay) ν_μ turning into ν_τ (transition rate into electron neutrinos ν_e assumed ignorable in this case):

$\Delta E^2 = 3 \times 10^{-3}\ eV^2$ $\theta = 45°$

Electron neutrinos ν_e from Sun tuning into ν_μ or ν_τ or a mix of them (the experiments where this is based are not sensitive enough to tell apart which is the case):

$\Delta E^2 = 1 \times 10^{-4}\ eV^2$ $\theta = 35°$

Here is denoted: $\Delta E = E_2 - E_1$

The full theory for 3 neutrinos

The transformation between the three-neutrino states is:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} P_{e1} & P_{e2} & P_{e3} \\ P_{\mu 1} & P_{\mu 2} & P_{\mu 3} \\ P_{\tau 1} & P_{\tau 2} & P_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{58}$$

The 3x3 unitary transformation matrix above is called Pontecorvo - Maki - Nakagawa - Sakata (PMNS) matrix. It gives the probability of a neutrino of given flavor α = e, μ, or τ to be found in a mass eigenstate j = 1, 2, or 3. These probabilities are proportional to the square modulus of matrix elements.

The 3-neutrino matrix in (58) is in matter of fact quite easy to develop from the 2-neutrino theory above. Define matrices that change a three neutrino state, but only two oft the neutrinos at a time. We get then three matrices, that all obey condition (7). Where a neutrino is not to be changed we put 1 on the diagonal, and other ways zeros on the corresponding row and column.

$$\mathbf{M} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{59}$$

$$\mathbf{N} = \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} \\ 0 & 1 & 0 \\ -\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \tag{60}$$

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \tag{61}$$

Then the corresponding 2-neutrino operation (in this case affecting only electron-neutrino and mu-neutrino) becomes:

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \nu_e \cos\theta_{12} + \nu_\mu \sin\theta_{12} \\ -\nu_e \sin\theta_{12} + \nu_\mu \cos\theta_{12} \\ \nu_\tau \end{pmatrix} \tag{62}$$

etc. Multiplicating the matrices gives the total transformation:

$$\mathbf{P} = \mathbf{S}\ \mathbf{N}\ \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} \\ 0 & 1 & 0 \\ -\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \dots \tag{63}$$

$$\dots = \begin{pmatrix} \cos\theta_{12}\ \cos\theta_{13} & \sin\theta_{12}\ \cos\theta_{13} & \sin\theta_{13} \\ -\sin\theta_{12}\ \cos\theta_{23} - \cos\theta_{12}\ \sin\theta_{23}\ \sin\theta_{13} & \cos\theta_{12}\ \cos\theta_{23} - \sin\theta_{12}\ \sin\theta_{23}\ \sin\theta_{13} & \sin\theta_{23}\ \cos\theta_{13} \\ \sin\theta_{12}\ \sin\theta_{23} - \cos\theta_{12}\ \cos\theta_{23}\ \sin\theta_{13} & -\cos\theta_{12}\ \sin\theta_{23} - \sin\theta_{12}\ \cos\theta_{23}\ \sin\theta_{13} & \cos\theta_{23}\ \cos\theta_{13} \end{pmatrix}$$

It is unitary because the multiplied matrices are unitary:

$$\mathbf{P}^T\ \mathbf{P} = (\mathbf{S}\ \mathbf{N}\ \mathbf{M})^T\ (\mathbf{S}\ \mathbf{N}\ \mathbf{M}) = \mathbf{M}^T\ \mathbf{N}^T\ \mathbf{S}^T\ \mathbf{S}\ \mathbf{N}\ \mathbf{M} = \mathbf{M}^T\ \mathbf{N}^T\ \mathbf{N}\ \mathbf{M} = \mathbf{M}^T\ \mathbf{M} = \mathbf{1} \tag{64}$$

The matrix (63) is almost the PMNS-matrix. In the standard representation of the PMNS matrix with angle parameters (corresponding to and in a way similar to the Euler-angle representation of 3-dimesional rotation matrix):

$$\mathbf{P'} = \begin{pmatrix} \cos\theta_{12}\ \cos\theta_{13} & \sin\theta_{12}\ \cos\theta_{13} & \sin\theta_{13}\ e^{i\delta} \\ -\sin\theta_{12}\ \cos\theta_{23} - \cos\theta_{12}\ \sin\theta_{23}\ \sin\theta_{13}\ e^{i\delta} & \cos\theta_{12}\ \cos\theta_{23} - \sin\theta_{12}\ \sin\theta_{23}\ \sin\theta_{13}\ e^{i\delta} & \sin\theta_{23}\ \cos\theta_{13} \\ \sin\theta_{12}\ \sin\theta_{23} - \cos\theta_{12}\ \cos\theta_{23}\ \sin\theta_{13}\ e^{i\delta} & -\cos\theta_{12}\ \sin\theta_{23} - \sin\theta_{12}\ \cos\theta_{23}\ \sin\theta_{13}\ e^{i\delta} & \cos\theta_{23}\ \cos\theta_{13} \end{pmatrix} \tag{65}$$

This can be constructed similarly as (63), the only difference is that the matrix $\mathbf{N}$ is replaced by:

$$\mathbf{N'} = \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}\ e^{i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}\ e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \tag{66}$$

This is unitary, and then also (61) is unitary:

$$\mathbf{P'}^{-1} = \mathbf{P'}^{\dagger} \tag{67}$$

Explanation for the $e^{i\delta}$ factor:

The phase factor angle δ is non-zero only if neutrino oscillations violate the CP-symmetry (charge+parity-mirroring symmetry), which symmetry weak interactions generally are found to violate. This is expected for neutrino oscillations too, but is not yet observed experimentally. Nevertheless, the factor is there in order to make the theory general enough to cover also that case.

Experimental results for the full theory

The three angles θ_{12} , θ_{13} , θ_{23} , and the phase factor angle δ that must be decided experimentally. From experimental results the "Euler"-angles (in degrees) are found to be approximately:

$\theta_{12} = 35\ °$ $\theta_{23} = 45°$ $\theta_{13} = 10\ °$

Note that the angles above correspond to two-neutrino cases where the indices give which two of the three neutrinos are picked up to the approximate two-neutrino model: index 12 means electron and muon neutrino, index 13 electron and taon neutrino, and index 23 muon and taon neutrino. One can get back the matrices (55), (56), and (57) from (60) by using one angle only and setting then the other angles = 0.

The mass (= energy) differences, that also are needed in the theory, are related as follows:

$$\Delta m_{31} = m_3 - m_1$$

$$\Delta m_{21} = m_2 - m_1$$

$$\Delta m_{32} = m_3 - m_2 = (m_3 - m_1) - (m_2 - m_1) = \Delta m_{31} - \Delta m_{21}$$

Experimental values:

$$\Delta m_{32}{}^2 = 0.003\ eV^2$$

$$\Delta m_{21}{}^2 = 0.0001\ eV^2$$

Appendices

Time evolution - formal treatment

The flavor states and the mass states, as was already told above, are not identical. The neutrino masses refer to the mass states alone, and not to the flavor sates. The flavor states, the three kind of neutrinos, ν_e, ν_μ, ν_τ, are not associated with any certain masses.

In stead, the flavor states are a mixes of the mass states. Vice versa, taking a certain mass state it is a mix of the flavor states. The crucial point is that the mixings are time dependent. It means, that if we initially (at zero time) detect only one flavor (say, electron neutrino), some time later detecting the same neutrino beam we will find a mix of them all three, and maybe in some later moment again only one but different flavor (say, mu neutrino), etc.

This oscillating time dependency can be expressed applying a time evolution operator to the initial state. Below a short formal treatment of the time evolution.

Flavor base states denoted: $| \nu_\alpha \rangle = | \nu_e \rangle, | \nu_\mu \rangle, | \nu_t \rangle$ (68)

Mass (and energy) eigenstates: $| E_j \rangle = | E_j \rangle, | E_j \rangle, | E_j \rangle$ (69)

$$| E_j \rangle = \sum_{\alpha} | \nu_\alpha \rangle \langle \nu_\alpha | E_j \rangle = \sum_{\alpha} \langle \nu_\alpha | E_j \rangle | \nu_\alpha \rangle = \sum_{\alpha} P_{\alpha j} | \nu_\alpha \rangle \tag{70}$$

were $P_{\alpha j}$ is a matrix element of the the Pontecorvo - Maki - Nakagawa - Sakata (PMNS) matrix.

Thus we can write in matrix operator form:

$$\left| E_j \right\rangle = \mathbf{P} \left| \nu_\alpha \right\rangle \tag{71}$$

and the matrix elements are given by:

$$P_{\alpha j} = \langle \nu_\alpha | E_j \rangle \tag{72}$$

Here E_j are the energy eigenvalues; $j = 1, 2, 3$.

We want to know how the neutrinos evolve in time, in other words, we search the time evolution operator for the neutrino flavor states:

$$\left| \nu_\alpha , t \right\rangle = \mathbf{U_f} (t) \left| \nu_\alpha \right\rangle \tag{73}$$

From a general theory of free particles with a mass we have the time evolution operator $\mathbf{U}(t)$. We can assume it be known, in any case of form, provided the masses are non-zero. In the base of the mass (energy) eigenstates it is represented by a diagonal matrix operator where the diagonal elements are given by:

$$U_{jj}(t) = e^{-i E_j t} \tag{74}$$

Time evolution of a mass eigenstate can then be expressed:

$$\left| E_j , t \right\rangle = \mathbf{U}(t) \left| E_j \right\rangle = \mathbf{U}(t) P \left| n_a \right\rangle \tag{75}$$

Transforming back to the neutrino flavor base gives the time evolution of the neutrino flavor states:

$$\left| n_a , t \right\rangle = \mathbf{P}^\dagger \left| E_j , t \right\rangle = \mathbf{P}^\dagger \mathbf{U}(t) \left| E_j \right\rangle = \mathbf{P}^\dagger \mathbf{U}(t) \mathbf{P} \left| n_a \right\rangle \tag{76}$$

The time evolution operator for neutrino flavor states thus is

$$\mathbf{U_f} (t) = \mathbf{P}^\dagger \mathbf{U}(t) \mathbf{P} \tag{77}$$

The question of the origin of the neutrino mass

It has been argued that the neutrino mass, as it is so extremely low, is not from the same source as the masses of the most of the other particles, the hypothetical Higgs field. It has been suggested that the neutrino mass in stead comes from a Majorana mass term, that has the effect preventing neutrinos from being Majorana particles, particles that are their own antiparticles.

One such hypothesis is called the Seesaw mechanism. All neutrinos still detected are of left-handed chirality. The old theory of massless neutrinos supports that view. But theoretically neutrinos with a mass can also appear with right-hand chirality (due to a Lorentz transformation). The Seesaw mechanism suggest the existence of right-hand neutrinos that are extremely massive, which for that reason would be difficult to detect.

There are other theories still. There is no scientific consensus of the subject yet to day (2012).

Currently ongoing neutrino experiments (2012)

Below listed a few of the more recent neutrino experiments related to the theory:
- Super Kamiokande, Japan
- K2K Long Baseline Neutrino Oscillation Experiment, Japan
- Neutrino Mass Experiment at the University of Mainz/Germany
- Katrin - a future large tritium beta decay experiment, Karlsruhe/Germany
- Troitsk Neutrino Mass Experiment , Institute for Nuclear Research, Moscow

Bibliography and background information:

There is no list of papers or books to refer on. This text was edited by me, mainly from educational material that is or has been available for quantum mechanics and particle physics courses in Stockholm University between years 2007 - 2010, and from material available in Internet, mainly from Wikipedia and CERN. If there are errors in the text or in the formulas they are most probably from the editors hand.

The method proving of the neutrino oscillation starting generally from a unitary transformation matrix is my own construction. I am aware that it must been shown before, although I am not in possession of any such paper.

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